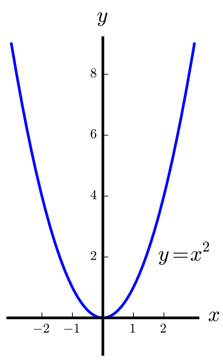
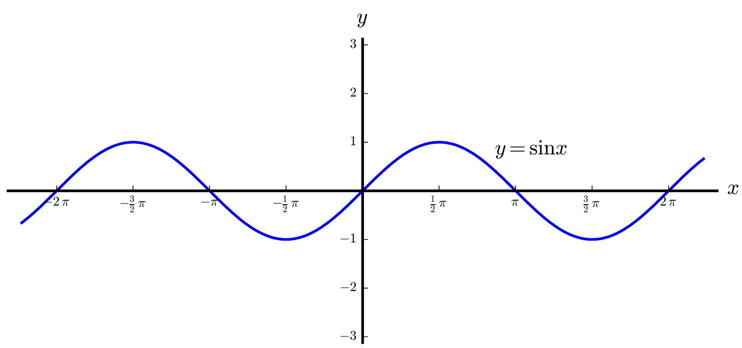
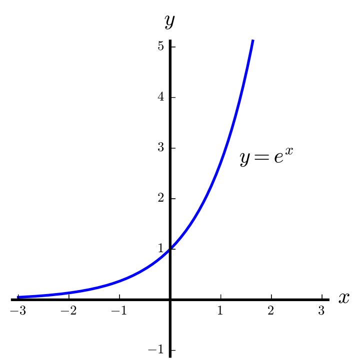
Precalculus Review

Summary and Exercises for Students Taking Calculus







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Version 1.1
September 2023

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Preface

The Purpose of These Notes

Learning calculus entails both grasping new concepts—which are within the reach of the vast majority of college students—and being able to demonstrate their conceptual understanding via the solution of exercises on homework and exams.

The actual ability of students to solve calculus exercises depends not only upon whether or not they understood the concepts discussed in a calculus class, but, no less important, upon having a solid background in precalculus. Students who are not fluent in precalculus often find themselves in the frustrating situation of understanding the ideas presented in a calculus class, but being unable to solve many of the homework and exam exercises, because these exercises make regular use of ideas and methods from precalculus.

Indeed, it might be said that in the majority of cases, the ability of a student to succeed in calculus is determined primarily by their comfort with precalculus.

The purpose of these notes is to provide a resource for those students who enroll in an introductory calculus class and who have taken a precalculus course but, for whatever reason, are not sufficiently proficient in precalculus at the level needed for success in the calculus class.

The material in these notes is presented as a review, aimed at students who have already studied precalculus, and as such the material is presented much more concisely than would be seen in a textbook aimed at teaching precalculus to students who had not previously seen that material.

Choice of Material

Not everything taught in precalculus is needed to succeed in calculus, and these notes cover only those aspects of precalculus that are routinely used in an introductory calculus class at the level typically taught in the United States.

The reader might observe that the first three sections of these notes, titled Numbers, Expressions and Equations, and Functions and Graphs—which treat topics that typically precede a precalculus course—are longer than some of the later sections, which are about precalculus per se. That is because, in the experience of the author, the problems encountered by students in an introductory calculus course involving the material in the first three sections of these notes often involve misunderstanding, which needs more attention to be corrected, whereas the problems involving the material from the later sections are often simply a matter of the students having forgotten (or not seen) the needed parts of this material, rather than outright misunderstanding.

Exercises

Mathematics is learned by actively doing it, not by passively reading about it. Certainly, listening to lectures and reading the textbook is an important part of any mathematics course, but the real learning occurs when the student does exercises, which provide an opportunity for the student to make use of the concepts discussed in class, and to see whether or not she understands these concepts. As such, the reader is strongly encouraged not only to read these notes, but to try various of the exercises. Indeed, focusing on the examples and exercises, and reading the text inasmuch as is needed to understand how to do the exercises, is the preferable approach.

Calculators and Computers

A calculator will not be needed for these notes, though a calculator can certainly be used for some simple numerical calculations (except in those exercises that explicitly say not to use a calculator). Free calculator apps are available for smartphones, tablets and computers. Programmable and/or graphing calculators are very expensive and are neither needed nor worth the cost; the reader who already owns such a calculator should feel free to use one it for numerical calculations, though using it for graphing and the like will in fact be a obstacle to developing fluency in precalculus.

Writing the Exercises

The writing of mathematics, and in particularly the writing of mathematics homework exercises of the sort that appear in these notes, is often, unfortunately, not given sufficient attention. As a result, many students develop the mistaken idea that writing mathematics assignments is simply a matter of jotting down some numbers and formulas, often in almost random order on the page. In fact, nothing could be further from the truth.

There are two main reasons why careful writing in mathematics, similarly to writing in all other fields, is crucially important: it allows us to clarify our own thinking and make sure that our ideas are correct in all details, and it allows us to communicate our ideas to others (both teachers and other students) so that they can understand what we meant. Even professional mathematicians will acknowledge that until they write their ideas properly, they cannot be sure if they are correct.

Sloppy writing—by which we mean not only illegible and unorganized writing on the page, but, much more importantly, writing without proper grammar and punctuation, and without a logical flow of ideas from start to finish—often reflects sloppy thinking, and is often very hard for the reader to follow. Both learning and doing mathematics seriously entails treating writing not as an afterthought but as a crucial tool that may entail extra effort, but more than repays the effort it requires.

Everyone makes honest mathematical mistakes, but there is no reason to get in your own way by writing your homework carelessly. Properly written mathematics homework assignments entail the following basic points.

1. Write your homework assignments neatly and clearly.
2. Distinguish between scratch work and the final draft. Expect to do scratch work on separate paper prior to writing the final draft.
3. Your final draft should stand on its own; check your solutions by reading them as if they were written by someone else.
4. For each problem, write every step of your calculation or argument, and do so in a logical order from beginning to end. Numerical answers without justification, or equations on the page in random order, are not be acceptable for the final draft.
5. Use verbal explanations whenever needed. Formulas and calculations are not always sufficient.

6. Be very careful with "=" signs. You must write "=" between things that are equal, and not write "=" between things that are not equal.

Errors and Suggestions

In spite of the author's best effort, there will inevitably be some errors in these notes. If the reader finds any such errors, or if the reader has any suggestions for improving these notes, it would be very helpful if they would send them to the author at `bloch@bard.edu`.

Updated Version of These Notes

These notes will be updated as the need arises. This version of the notes is from September 2023. To see if there is a newer version of the notes, please check the webpage `http://faculty.bard.edu/bloch/publications/`.

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1 Numbers

Calculus makes use of precalculus—hence the name of the latter—but to do precalculus, a solid knowledge of basic algebra is needed. Algebra, in turn, is based upon numbers (of various types) and the operations we use to combine numbers. We review here a few of the most important ideas about numbers that are needed for both precalculus and calculus.

Types of Numbers

Precalculus and calculus take place within the context of the real numbers. Among the real numbers, there are some import special types of numbers that are frequently used in mathematics.

Types of Numbers

1. The **real numbers**, denoted $\mathbb{R}$, are all the numbers on the number line, including positive numbers, negative numbers, zero, whole numbers, fractions, and all other numbers (such as $\sqrt{2}$ and π).
2. The **rational numbers**, denoted $\mathbb{Q}$, are all numbers that are expressible as fractions, for example $\frac{2}{3}$ or -0.5.
3. The **integers**, denoted $\mathbb{Z}$, are the numbers $\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$.
4. The **natural numbers** (also called the **positive integers**), denoted $\mathbb{N}$, are the numbers $1, 2, 3, 4, \ldots$.

Note that all natural numbers are integers, and all integers are rational numbers, and all rational numbers are real numbers, but not the other way around.

There is a widely used set of numbers that is even larger than the set of real numbers, namely, the set of complex numbers, denoted $\mathbb{C}$. The complex numbers are not typically used in *Calculus I* and *Calculus II*, and as such are not discussed in these notes; they do arise in some subsequent mathematics courses, for example differential equations.

Properly Writing Multiplication

All students know that the correct way to write the addition and subtraction of two numbers is, for example, to write $5 + 2$ and $5 - 2$. The same goes for addition and subtraction involving symbols, for example $x + 3$

and $x - 3$.

By contrast, a very common mistake is in the way the multiplication of numbers and symbols is written; this mistake is unfortunately found in some published high school and college textbooks.

For example, suppose we want to write 5 times 3. Two common—and definitely incorrect—ways of writing this product are

$$5(3) \quad \text{and} \quad 5 \times 3.$$ ✗ **Incorrect Writing**

The expression "5(3)" is problematic because it leads many students to think that parentheses denote multiplication, which is not what parentheses mean, and this incorrect use of parentheses leads many students not to use parentheses correctly for when they are truly needed, which is for grouping; we will see an example of that shortly.

The expression "5×3" is problematic because the $\times$ symbol when written by hand, looks very much like the letter x when written by hand, and that leads to confusion, because in precalculus and calculus we often use the letter x as a variable, and we do not want any confusion about what is an x and what is a $\times$.

There is one and only one way to write 5 times 3, and that is

$$5 \cdot 3.$$ ✓ **Good Writing**

When we write multiplication using letters, we can either write it with the $\cdot$ symbol, for example $5 \cdot x$ or $a \cdot b$, but we can also drop the $\cdot$ symbol in these cases and write $5x$ and ab, where adjacency denotes multiplication. Of course, we cannot drop the $\cdot$ symbol from $5 \cdot 3$, because then it would look like the number 53.

☞ **Error Warning** Never use $\times$ or () to denote multiplication. Always use the $\cdot$ symbol to denote multiplication, or use adjacency when letters are involved.

Order of Operations and Parentheses

Whereas parentheses should not be used to denote multiplication, they are crucial when maintaining the correct order of operations. The failure to write mathematical expressions properly in regard to the order of operations is a common error by calculus students.

The order of operation is as follows.

Order of Operations

1. The order of numerical operations in algebra, from first to last, is
   (1) Parentheses,
   (2) Exponents,
   (3) Multiplication,
   (4) Division,
   (5) Addition,
   (6) Subtraction.
2. A mnemonic for the order of operations in English is PEMDAS, which is made up of the first letters of the operations in the correct order.

For example, suppose we want to write 5 + 2 times 3. Two common—and definitely incorrect—ways of writing this product are

$$5 + 2\cdot 3 \quad \text{and} \quad 5 + 2(3).$$ ✗ **Incorrect Writing**

The expression "$5 + 2\cdot 3$" is a valid mathematical expression, but it simply does not mean 5 + 2 times 3, because multiplication comes before addition in the order of operations, and so this expression means first multiplying 2 times 3, and only then adding 5, which yields.

$$5 + 2\cdot 3 = 5 + 6 = 11.$$

The expression "$5 + 2(3)$" is even worse, because not only does it have the same mistake as above regarding the order of operations, but it also involves an incorrect use of parentheses. The origin of this type of error when attempting to write 5+2 times 3 is the misunderstanding of parentheses mentioned above, namely, the mistaken idea that parentheses denote multiplication—which is incorrect but often taught. The problem with using parentheses to denote multiplication is that students who use parentheses for that incorrect purpose are likely not to use parentheses where they are in fact needed, which is for grouping to overcome the order of operations.

The correct way to write 5 + 2 times 3 is

$$(5 + 2)\cdot 3,$$ ✓ **Good Writing**

where the parentheses are used to overcome the standard order of operations and force us to do the addition first. The result of this calculation is

$$(5 + 2)\cdot 3 = 7\cdot 3 = 21,$$

which shows that we previously obtained the incorrect answer when we did not use the needed parentheses.

If we think about parentheses correctly—for grouping and not multiplication—then we will know to write 5 + 2 times 3 correctly as $(5 + 2)\cdot 3$, where the parentheses are indeed used for grouping, and not as "$5 + 2(3)$," which gives the wrong outcome.

☞ **Error Warning** Always use () to group expressions when doing multiplication.

Here is an example of using parentheses.

Example 1

Simplify each of the following expressions.

(a) $(5x - 2)x + 3$.

(b) $5x - 2(x + 3)$.

(c) $(5x - 2)(x + 3)$.

Solution

(a) Here $5x - 2$ is multiplied by only x. We compute

$$(5x - 2)x + 3 = 5x\cdot x - 2\cdot x + 3 = 5x^2 - 2x + 3.$$

(b) Here only 2 is multiplied by $(x+3)$. We compute

$$5x - 2(x+3) = 5x - 2\cdot x - 2\cdot 3 = 5x - 2x - 6 = 3x - 6.$$

(c) Here $5x-2$ is multiplied by $x+3$. We compute

$$(5x-2)(x+3) = 5x\cdot x + 5x\cdot 3 - 2\cdot x - 2\cdot 3 = 5x^2 + 15x - 2x - 6 = 5x^2 + 13x - 6.$$

Equals Signs

Mathematics makes extensive use of symbols, and one of the most important mathematical symbols—and arguably the most important of them all—is the "=" sign. Though the idea of two numbers or formulas being equal seems straightforward, students in mathematics classes often use equals signs incorrectly, leading both to mathematical errors and to correct results that are written in a way that is very difficult for anyone other than the writer to follow.

The "=" sign in mathematics means *literally equal*, and it should be used whenever two things are precisely equal, and it should not be used when two things are related in some way but are not literally equal. Common mistakes involve both not writing the "=" sign when needed and writing the "=" sign when it is not correct to do so; both mistakes are equally problematic.

For example, as we will see in Chapter 8, suppose we are asked to simplify the expressions $5a^4b^{-3}(2a^{-2}b^2)^5$. A common—and definitely incorrect—way of writing the answer is

$$\begin{gathered} 6a^4b^{-3}(2ab^2)^5 \\ 6a^4b^{-3}2^5(a^{-2})^5(b^2)^5 \\ 6a^4b^{-3}32a^{-10}b^{10} \\ 6\cdot 32a^4a^{-10}b^{-3}b^{10} \\ 192a^{-6}b^7. \end{gathered}$$

✗ **Incorrect Writing**

The problem here is that various expressions are written without any indication of what equals what. The writer of the above expressions might intend to say that these expressions are all equal to each other, but in writing mathematics there is no room for ambiguity, and when the writer means to say that certain expressions are equal, that should be stated explicitly via the use of equals signs. The correct way to write the above simplification would be

$$\begin{aligned} 6a^4b^{-3}(2ab^2)^5 &= 6a^4b^{-3}2^5(a^{-2})^5(b^2)^5 \\ &= 6a^4b^{-3}32a^{-10}b^{10} \\ &= 6\cdot 32a^4a^{-10}b^{-3}b^{10} \\ &= 192a^{-6}b^7. \end{aligned}$$

✓ **Good Writing**

One way to understand the proper use of "=" signs is to think of the above calculation as a sentence, where "$6a^4b^{-3}(2ab^2)^5$" is the subject, and "=" is the verb. Writing mathematics should be done just as carefully, including following the rules of grammar, as any other writing, because careful writing helps the writer think carefully and the reader understand the intentions of the writer.

☞ **Error Warning** Always write "=" signs between expressions that are literally equal.

Another common—and definitely incorrect—way of writing the above calculation is

$$\begin{array}{c} 6a^4b^{-3}(2ab^2)^5 \quad \Rightarrow \quad 6a^4b^{-3}2^5(a^{-2})^5(b^2)^5 \\ \text{therefore} \quad 6a^4b^{-3}32a^{-10}b^{10} \\ \therefore \quad 6\cdot 32a^4a^{-10}b^{-3}b^{10} \quad \text{so} \quad 192a^{-6}b^7. \end{array}$$

✗ Incorrect Writing

The problem here is that various symbols and words other than "=" signs are used to mean what ought to be written using "=" signs. The "=" sign is the only symbol that means literally equal, and nothing else should be used instead.

> ☞ **Error Warning** Do not use symbols or words other than the "=" sign to indicate equality.

Yet another common type of incorrect writing of mathematical calculations involves a variant of the above simplification of the expression $6a^4b^{-3}(2ab^2)^5$. Suppose that instead of being told to simplify the expression, the assignment had been to verify that $6a^4b^{-3}(2ab^2)^5 = 192a^{-6}b^7$. A common—and again definitely incorrect—way of writing the answer is

$$\begin{aligned} 6a^4b^{-3}(2ab^2)^5 &= 192a^{-6}b^7 \\ 6a^4b^{-3}2^5(a^{-2})^5(b^2)^5 &= 192a^{-6}b^7 \\ 6a^4b^{-3}32a^{-10}b^{10} &= 192a^{-6}b^7 \\ 6\cdot 32a^4a^{-10}b^{-3}b^{10} &= 192a^{-6}b^7 \\ 192a^{-6}b^7 &= 192a^{-6}b^7. \quad \checkmark \end{aligned}$$

✗ Incorrect Writing

The checkmark is meant to indicate that the calculation had reached a true statement, and as such it is complete. The problem here is that the logic is entirely backwards, in that the first line of the calculation assumes the thing we are trying to show, and the end of the calculation deduces something that is obviously true. The correct way to write this calculation is exactly as we wrote it above, which started with one side of the equation we are trying to show, namely $6a^4b^{-3}(2ab^2)^5$, and showing, via a sequence of equalities, that it equals the other side of the equation we are trying to show, namely $192a^{-6}b^7$.

> ☞ **Error Warning** Never try to show that two things are equal by assuming that they are equal, and deducing something from that. Showing that something is true can never involve assuming that it is true.

In addition to the problem of not writing "=" signs to indicate equality, there is the opposite problem of writing "=" signs to denote something other than equality.

For example, suppose we want to find the solution of the equation $x^3 = 8$. A common—and definitely incorrect—way of writing the answer is

$$x^3 = 8 = x = 2.$$

✗ Incorrect Writing

The problem here is that what is written literally says that 8 equals 2, which, of course, is not correct. The "=" sign means literal equality, and it should not be used for anything other than literal equality; it should not be used to mean that two things are somehow related other than by being equal, and it should not be used to replace words such as "therefore," "hence" and the like. The correct way to write the above solution would be

$$x^3 = 8 \quad \text{therefore} \quad x = 2.$$

✓ Good Writing

☞ **Error Warning** Do not use the "=" sign to indicate anything other than literal equality.

The two most important features of mathematical writing are clarity and accuracy. Writing "=" signs between expressions that are equal makes the writing more clear, whereas writing "=" signs between expressions that are not literally equal ruins accuracy. In particular, numbers being approximately equal is not the same as the numbers being literally equal, and a common instance of inaccuracy in mathematical writing is to write an "=" sign between expressions that are only approximately equal. For example, a common—and definitely incorrect—way of writing the approximate value of π is

$$\pi = 3.14159,$$ ✗ **Incorrect Writing**

which is not correct because π has infinitely many (and not repeating) decimals. The correct way to write this approximation of π is to use a "≈" sign, which denotes approximately equal but not exactly equal. For example, we would correctly write

$$\pi \approx 3.14159.$$ ✓ **Good Writing**

☞ **Error Warning** Always write an "≈" sign between numbers that are approximately equal but not exactly equal.

Finally, one confusing use of "=" signs is when they are used instead of words as part of a sentence. For example, a common—and definitely incorrect—way of writing with "=" signs is

the height of the person = 6 feet, ✗ **Incorrect Writing**

which is not correct because what is actually written says that the word "person" equals the number 6. The problem is that whereas the height of the person literally equals 6 feet, an "=" sign can be used to denote that equality only between two symbols, and the "=" sign in the above incorrect sentence is not meant literally but rather is being used as a replacement for the word "is." The correct way to write the above sentence is

the height of the person is 6 feet. ✓ **Good Writing**

Alternatively, if symbols are preferred, it would also be correct to write

if h is the height of the person, then $h = 6$ feet. ✓ **Good Writing**

☞ **Error Warning** Do not use "=" signs as an abbreviation for words in a sentence. Do not write that a word equals a number.

Infinity

We will, at times, be using the symbols ∞ and $-\infty$ to denote "infinity" and "negative infinity," respectively. These words are written in quotes to emphasize the fact that these symbols do not denote numbers. Rather, these symbols represent what happens as we take numbers that get larger and larger without bound (going to ∞), and as we take numbers that get smaller and smaller without bound (going to $-\infty$), which means negative numbers having larger and larger magnitude.

For example, the numbers $2, 4, 8, 16, 32, \ldots$ are "going to ∞," and the numbers $-1, -3, -5, -7, -9, \ldots$ are "going to $-\infty$." It is important to note, however, that these numbers never "reach" ∞ or $-\infty$, because there are no numbers there to reach.

Given that ∞ and $-\infty$ are not numbers, we cannot do numerical calculations with them. For example, it is not correct to do calculations such as "$\infty + 5 = \infty$" or "$\infty - \infty = 0$" as if ∞ could be added or subtracted the way we add numbers.

☞ **Error Warning** The symbols ∞ and $-\infty$ are not numbers. They cannot be added, subtracted, multiplied and divided as we do with numbers; the standard rules of arithmetic do not apply to the symbols ∞ and $-\infty$.

Intervals

Intervals are a very useful type of sets of real numbers. An interval is the set of all numbers between two fixed numbers, where the endpoints might or might not be included in the interval. The different types of interval, where the variety comes from the endpoints, are as follows.

Intervals

Let a and b be real numbers. Suppose that $a \leq b$.

Notation	Type of Interval	Definition
(a, b)	**open bounded interval**	$a < x < b$
$[a, b]$	**closed bounded interval**	$a \leq x \leq b$
$[a, b)$	**half-open interval**	$a \leq x < b$
$(a, b]$	**half-open interval**	$a < x \leq b$
(a, ∞)	**open unbounded interval**	$a < x$
$(-\infty, b)$	**open unbounded interval**	$x < b$
$(-\infty, \infty)$	**open unbounded interval**	all real numbers
$[a, \infty)$	**closed unbounded interval**	$a \leq x$
$(-\infty, b]$	**closed unbounded interval**	$x \leq b$

For example, the interval $[2, 5]$ is the set of all real numbers x such that $2 \leq x \leq 5$. The interval $(3, \infty)$ is the set of all real numbers x such that $3 < x$.

Observe that in the list of intervals, there are no intervals of the form "$(a, \infty]$," or "$[a, \infty]$," or "$[-\infty, b)$" or "$[-\infty, b]$," because the square bracket "[" and "]" in interval notation indicates that the endpoint is included in the interval, but the symbols ∞ and $-\infty$ are not numbers, and hence cannot be included as endpoints of intervals. Rather, the symbols ∞ and $-\infty$ are used in interval notation to indicate that the interval "goes on forever" in that direction.

☞ **Error Warning** The symbols ∞ and $-\infty$ are not numbers, and cannot be included in an interval. Hence, there are no intervals of the form "$(a, \infty]$," or "$[a, \infty]$," or "$[-\infty, b)$" or "$[-\infty, b]$."

There is a confusing aspect regarding the notation for open bounded intervals (a, b), for example $(1, 6)$, which is that this same notation is also used to mean some other things in mathematics. When discussing intervals, the notation $(1, 6)$ means the interval from 1 to 6, not including the endpoints. On the other hand, when discussing points in the plane (usually denoted $\mathbb{R}^2$), the notation $(1, 6)$ means the point in $\mathbb{R}^2$ with x-coordinate 1 and y-coordinate 6. (There is another common use of this notation in number theory, but that use is not typically found in a calculus class, so we will not discuss it.) The fact that the same mathematical notation can have very different meanings in different contexts can be confusing, but it is a historical accident with which we are now stuck. Fortunately, the meaning of the notation (a, b) can usually be figured out from the context.

☞ **Error Warning** The notation (a, b) can mean either the open bounded interval with endpoints a and b, that is, the set of all real numbers x such that $a < x < b$, or it can mean the point in the plane with x-coordinate a and y-coordinate b; the meaning of notation (a, b) is determined by the context.

Here are two examples of using intervals.

Example 2

Rewrite each of the following sets of real numbers as an interval.

(a) The set of all real numbers x such that $-3 < x \leq 10$.

(b) The set of all real numbers w such that $4 \leq w$.

Solution

(a) This set is the interval $(-3, 10]$.

(b) This set is the interval $[4, \infty)$.

Example 3

Rewrite each the following intervals as a set of real numbers defined by inequalities.

(a) $[\frac{2}{3}, 65)$

(b) $(-\infty, -9)$

Solution

(a) The set of all real numbers x such that $\frac{2}{3} \leq x < 65$.

(b) The set of all real numbers x such that $x < 9$.

Absolute Value

Numbers can be positive, negative or zero, and in some situations it is convenient to convert negative numbers to positive by "removing the negative sign," and leaving positive numbers and zero unchanged. The function that does that is the absolute value function, which is defined as follows.

Absolute Value

Let x be a real number. The **absolute value** of x, denoted $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

The idea of the above definition is that is a number x is negative, that is, if $x < 0$, then $-x > 0$, and so $-x$ is the same as "removing the negative sign" from x.

Here is an example of using absolute value.

Example 4

Find each of the following absolute values.

(a) $|3|$

(b) $|-15|$

(c) $|0|$

(d) $|x^2 + 4|$

Solution

(a) Because $3 \geq 0$, we have $|3| = 3$.

(b) Because $-15 < 0$, we have $|-15| = -(-15) = 15$.

(c) Because $0 \geq 0$, we have $|0| = 0$.

(d) For any real number x, we know that $x^2 \geq 0$, and so $x^2 + 4 > 0$, which implies that $|x^2 + 4| = x^2 + 4$.

The absolute value function has a number of useful properties, including the following.

Properties of Absolute Value

Let a, b and r be real numbers.

1. $|-a| = |a|$.
2. $|a|^2 = a^2$.
3. $|a - b| = |b - a|$.
4. $|ab| = |a|\,|b|$.

The the above properties of absolute value can be applied not only to actual numbers, but to any expressions that have numerical values. For example, we see that $|-3x^5| = |3x^5|$ and $|7x - 2y| = |2y - 7x|$.

EXERCISES – Chapter 1

1–4 ■ Rewrite each of the following sets of real numbers as an interval.

1 . The set of all real numbers x such that $2 < x < 5$

2 . The set of all real numbers p such that $-6 < p \leq 13$

3 . The set of all real numbers y such that $-\frac{1}{2} \leq y < -\frac{1}{7}$

4 . The set of all real numbers m such that $2 \leq m \leq 5$

5–8 ■ Rewrite each of the following sets of real numbers as an interval.

5 . The set of all real numbers z such that $-32 < z$

6 . The set of all real numbers x such that $4.287 \leq x$

7 . The set of all real numbers t such that $t < \frac{5}{6}$

8 . The set of all real numbers w such that $w \leq -15$

9–16 ■ Rewrite each the following intervals as a set of real numbers defined by inequalities.

9 . $[2, 5)$

10 . $[-0.3, 6]$

11 . $(-5, \frac{8}{3})$

12 . $(-12, -3]$

13 . $[2, \infty)$

14 . $(-\infty, 2.57)$

15 . $(-\infty, -8]$

16 . $(\frac{3}{14}, \infty)$

2 Expressions, Equations and Inequalities

We continue here our review of algebra, now focusing on expressions, equations and inequalities. It is important to distinguish between expressions on the one hand, and equations and inequalities on the other hand.

An expression in our present context is a collection of numbers and letters combined by the usual operations of arithmetic. Some examples of the sort of expressions that might be encountered in calculus are

$$x^2 + 3x + 7 \quad \text{and} \quad \frac{\sqrt{x+h} - \sqrt{x}}{h} \quad \text{and} \quad \frac{\frac{6}{x+1} - \frac{3}{x-3}}{\frac{8x}{x-4} - \frac{5x}{2}}.$$

Observe that expressions of this sort do not contain "=" signs, or "<" or "≤" signs, and they cannot be "solved." What we can do with expressions is to simplify them, or factor them, or rearrange them, which we do by showing that the original expression is equal to some simpler (or otherwise preferable) version of it.

Equations, by contrast, always do involve "=" signs. Specifically, an equation is a statement of equality of two expressions, where one expression is found on each side of the "=" sign. Some examples of the sort of equations that might be encountered in calculus are

$$x^2 + 3x + 7 = 0 \quad \text{and} \quad \frac{3}{x+5} - \frac{2}{x-1} = 2 \quad \text{and} \quad \ln(x^5) = 9 - \ln(2x).$$

When an equation contains an unknown—often denoted by a letter such as "x" or "y"—the main thing we do with such an equation is to solve for the unknown. We often solve an equation by showing that the equation implies some simpler (or otherwise preferable) version of it, though this new equation is not itself "equal" to the original equation.

Observe that we use the term "unknown" rather than "variable" when referring to the symbol "x" in an equation such as $x^2 + 3x + 7 = 0$. The reason that the term "unknown" is preferable is that nothing in an equation is varying, and a symbol such as "x" simply refers to a number, the value of which we do not yet know. Of course, we can have equations involving any letter as the unknown, not just x, for example the equation $3r + 2 = r^2 - 8$.

Finally, inequalities are similar to equations, though with "<" or "≤" signs instead of "=" signs. Specifically, an inequality is a statement saying that the numerical value of one expression is less than, or less than or equal to, the numerical value of another expression; as is the case with equations, so too with inequalities there is often an unknown, for which we want to solve.

Basic Algebra Formulas

In order to simplify the types of expressions we encounter in calculus, we make use of a few basic algebra formulas involving multiplying and factoring simple polynomials. These basic formulas are well worth memorizing.

Basic Algebra Formulas

Let a and b be real numbers.

1. $(a+b)^2 = a^2 + 2ab + b^2$.

2. $(a-b)^2 = a^2 - 2ab + b^2$.

3. $(a+b)(a-b) = a^2 - b^2$.

Here are two examples of multiplying and factoring using the basic algebra formulas stated above.

Example 1

Multiply and then simplify each of the following expressions by using basic algebra formulas.

(a) $(x+3y)^2$

(b) $(5r-4t^3)^2$

(c) $(2w^5-8)(2w^5+8)$

Solution

(a) We use the formula $(a+b)^2 = a^2+2ab+b^2$ with $a = x$ and $b = 3y$, and we compute

$$(x+3y)^2 = x^2 + 2\cdot x\cdot 3y + (3y)^2 = x^2 + 6xy + 9y^2.$$

(b) We use the formula $(a-b)^2 = a^2-2ab+b^2$ with $a = 5r$ and $b = 4t^3$, and we compute

$$(5r-4t^3)^2 = (5r)^2 - 2\cdot 5r\cdot 4t^3 + (4t^3)^2 = 25r^2 - 40rt^3 + 16t^6.$$

(c) We use the formula $(a+b)(a-b) = a^2-b^2$ (where the order of $a+b$ and $a-b$ can be reversed) with $a = 2w^5$ and $b = 8$, and we compute

$$(2w^5-8)(2w^5+8) = (2w^5)^2 - 8^2 = 4w^{10} - 64.$$

Example 2

Factor each of the following expressions by using basic algebra formulas.

(a) $36x^2+48x+16$

(b) $4t^2-4ts^4+s^8$

(c) $25p^2-9q^{12}$

Solution

(a) We use the formula $(a+b)^2 = a^2 + 2ab + b^2$ with $a = 6x$ and $b = 4$, and we compute

$$36x^2 + 48x + 16 = (6x)^2 + 2\cdot 6x\cdot 4 + 4^2 = (6x+4)^2.$$

(b) We use the formula $(a-b)^2 = a^2 - 2ab + b^2$ with $a = 2t$ and $b = s^4$, and we compute

$$4t^2 - 4ts^4 + s^8 = (2t)^2 - 2\cdot 2t\cdot s^4 + (s^4)^2 = (2t - s^4)^2.$$

(c) We use the formula $(a+b)(a-b) = a^2 - b^2$ with $a = 5p$ and $b = 3q^6$, and we compute

$$25p^2 - 9q^{12} = (5p)^2 - (3q^6)^2 = (5p + 3q^6)(5p - 3q^6).$$

There are also formulas for expressions such as $(a+b)^3$ that are useful on occasion, though there is no need to remember such formulas, because they can be looked up, or worked out as needed. For example, the expression $(a+b)^3$ can be computed by rewriting it as $(a+b)^2(a+b)$, using the formula for $(a+b)^2$, and multiplying the resulting polynomials. By contrast, the three basic algebra formulas listed above are so commonly used that looking them up every time they are needed would be an impediment to solving calculus exercises.

Fractions and Rational Expressions

A very common type of expression that needs to be simplified in calculus is a rational expression, that is, an expression in the form of a fraction, though it can have letters as well as numbers in each of the numerator and denominator. Some examples of the sort of rational expressions that might be encountered in calculus are

$$\frac{6}{x+1} \quad \text{and} \quad \frac{2x^6 + x^5 + 6x^4 + 11x^2 - x + 3}{x^5 + 2x^3 + x} \quad \text{and} \quad \frac{\frac{1}{3(x+h)-2} - \frac{1}{3x-2}}{h}$$

Simplifying a rational expression is done exactly the same as simplifying fractions involving numbers. Unfortunately (perhaps due in part to the widespread use of calculators in high school, which are more suited to decimals than fractions), some students taking calculus are not sufficiently comfortable with manipulating fractions. Specifically, for calculus we need to add, subtract, multiply and divide fractions that involve letters as well as numbers, and fractions that have fractions in their numerators and/or denominators.

One of the key things to keep in mind in algebra is that letters simply stand for numbers, the values of which we do not know, and we therefore treat letters exactly the same as we would treat numbers. In particular, the familiar rules for adding, subtracting, multiplying and dividing fractions with numbers work exactly the same for fractions with letters, and for built-up fractions.

We will not review here all the main methods used for manipulating fractions, such as adding and subtracting fractions; rather, we will focus on the aspect of fractions that causes the most trouble to calculus students, which is built-up fractions, that is, fractions that have fractions in their numerators and/or denominators (all the more so when such fractions have letters in addition to numbers).

For built-up fractions, it is important to distinguish the main fraction line from the subsidiary fraction lines. Visually, the best way to make this distinction is to write the main fraction line longer than the other fraction lines. Even better, the main fraction line should also be written level with the "=" sign.

There are three particular types of built-up fractions that we now examine, starting with numerical examples. In the following example, we repeatedly use the standard trick that any number c can be replaced by the fraction $\frac{c}{1}$.

Example 3

Simplify each of the following built-up fractions.

(a) $\dfrac{\frac{2}{3}}{7}$

(b) $\dfrac{2}{\frac{3}{7}}$

(c) $\dfrac{\frac{2}{3}+\frac{5}{7}}{9}$

Solution

(a) We compute

$$\frac{\frac{2}{3}}{7} = \frac{\frac{2}{3}}{\frac{7}{1}} = \frac{2}{3} \cdot \frac{1}{7} = \frac{2\cdot 1}{3\cdot 7} = \frac{2}{21}.$$

(b) We compute

$$\frac{2}{\frac{3}{7}} = \frac{\frac{2}{1}}{\frac{3}{7}} = \frac{2}{1} \cdot \frac{7}{3} = \frac{2\cdot 7}{1\cdot 3} = \frac{14}{3}.$$

(c) We compute

$$\frac{\frac{2}{3}+\frac{5}{7}}{9} = \frac{\frac{2\cdot 7+3\cdot 5}{3\cdot 7}}{9} = \frac{\frac{14+15}{21}}{\frac{9}{1}} = \frac{\frac{29}{21}}{\frac{9}{1}} = \frac{29}{21} \cdot \frac{1}{9} = \frac{29\cdot 1}{21\cdot 9} = \frac{29}{189}.$$

We now redo the above types of simplifications of built-up fraction with letters rather than numbers; these three simplifications should *not* be memorized, and are simply meant to show the general methods being used.

1. Simplify $\dfrac{\frac{a}{b}}{c}$.

   This fraction can be simplified by rewriting the denominator as $\frac{c}{1}$, yielding

$$\frac{\frac{a}{b}}{c} = \frac{\frac{a}{b}}{\frac{c}{1}} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}.$$

2. Simplify $\dfrac{a}{\frac{b}{c}}$.

   This fraction can be simplified by rewriting the numerator as $\frac{a}{1}$, yielding

$$\frac{a}{\frac{b}{c}} = \frac{\frac{a}{1}}{\frac{b}{c}} = \frac{a}{1} \cdot \frac{c}{b} = \frac{ac}{b}.$$

3. Simplify $\dfrac{\frac{a}{b}+\frac{c}{d}}{e}$.

   This fraction can be simplified by first adding the two fractions in the numerator, and then using the method of Item 1, yielding

$$\frac{\frac{a}{b}+\frac{c}{d}}{e} = \frac{\frac{ad+bc}{bd}}{e} = \frac{\frac{ad+bc}{bd}}{\frac{e}{1}} = \frac{ad+bc}{bd} \cdot \frac{1}{e} = \frac{ad+bc}{bde}.$$

The following two examples, which use the above ideas, are needed for calculus.

Example 4

Simplify the expression $\dfrac{\frac{1}{x+h} - \frac{1}{x}}{h}$.

Solution We compute

$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \frac{\frac{-h}{x(x+h)}}{\frac{h}{1}} = \frac{-h}{x(x+h)} \cdot \frac{1}{h} = -\frac{1}{x(x+h)}.$$

Example 5

Simplify the expression $\dfrac{\frac{x}{2x-7} + \frac{2}{x-9}}{\frac{x-8}{x-3} + \frac{4}{x}} = 0.$

Solution We compute

$$\begin{aligned}
\frac{\frac{x}{2x-7} + \frac{2}{x-9}}{\frac{x-8}{x-3} + \frac{4}{x}} &= \frac{\frac{x(x-9)+2(2x-7)}{(2x-7)(x-9)}}{\frac{x(x-8)+4(x-3)}{x(x-3)}} \\
&= \frac{\frac{x^2-9x+4x-14}{(2x-7)(x-9)}}{\frac{x^2-8x+4x-12}{x(x-3)}} \\
&= \frac{\frac{x^2-5x-14}{(2x-7)(x-9)}}{\frac{x^2-4x-12}{x(x-3)}} \\
&= \frac{\frac{(x-7)(x+2)}{(2x-7)(x-9)}}{\frac{(x-6)(x+2)}{x(x-3)}} \\
&= \frac{(x-7)(x+2)}{(2x-7)(x-9)} \cdot \frac{x(x-3)}{(x-6)(x+2)} \\
&= \frac{x(x-7)(x-3)}{(2x-7)(x-9)(x-6)}.
\end{aligned}$$

Example 6

Simplify the expression $\dfrac{\sqrt{x+h} - \sqrt{x}}{h}$.

Solution Here we use a little trick that is based upon the formula $(a+b)(a-b) = a^2 - b^2$. We compute

$$\begin{aligned}\frac{\sqrt{x+h}-\sqrt{x}}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{h}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}.\end{aligned}$$

The expression $\sqrt{x+h}+\sqrt{x}$ in the above example, which is used in order to remove the square roots in the numerator of the original expression, is referred to as the "conjugate" of $\sqrt{x+h}-\sqrt{x}$. In general, if we have an expression of the form $a-b$, its conjugate is $a+b$, and similarly the conjugate of $a+b$ is $a-b$. (There is a related concept of conjugate for complex numbers, but we are not using that here.)

Solutions of Equations

Before discussing how we solve equations, we first need to clarify what it means to have found a solution of an equation, because—though this point is often not sufficiently stressed—there is a difference between having a solution of an equation and finding a solution of an equation.

A solution of an equation is simply a value (in our case a numerical value) that when substituted into the equation makes the equation true. That a number is or is not a solution of an equation makes no reference to how that number might have been found.

Solution of Equation

Let $f(x) = g(x)$ be an equation with unknown x.

1. A **solution** of this equation is a number c such that replacing x with c in the equation, meaning $f(c) = g(c)$, yields a true statement.
2. An equation can have one solution, or more than one solution or no solution.

Here is an example of verifying whether or not a number is a solution of an equation.

Example 7

Verify whether or not each of the following numbers is a solution of the equation $x^3 - 13x = 3x^2 - 15$.

(a) $x = 2$

(b) $x = -3$

Solution

(a) If we substitute $x = 2$ into the left-hand side of the equation, we obtain

$$x^3 - 13x = 2^3 - 13\cdot 2 = 8 - 26 = -18,$$

and if we substitute $x = 2$ into the right-hand side of the equation, we obtain

$$3x^2 - 15 = 3\cdot 2^2 - 15 = 3\cdot 4 - 15 = 12 - 15 = -3.$$

Because the two sides of the equation are not equal when we substituted in $x = 2$, then $x = 2$ is not a solution of the equation.

(b) If we substitute $x = -3$ into the left-hand side of the equation, we obtain

$$x^3 - 13x = (-3)^3 - 13\cdot(-3) = -27 + 39 = 12,$$

and if we substitute $x = -3$ into the right-hand side of the equation, we obtain

$$3x^2 - 15 = 3\cdot(-3)^2 - 15 = 3\cdot 9 - 15 = 27 - 15 = 12.$$

Because the two sides of the equation are equal when we substituted in $x = -3$, then $x = -3$ is a solution of the equation.

Observe that when the number -3 was substituted into each side of the equation, it was written inside parentheses, which must always be done so that the negative is raised to the power to which x is raised.

Now that we know what it means for a number to be a solution of an equation, the next question is how we find a solution, and even better find all the solutions, of an equation, or find that there are no solutions. The process of finding the solution or solutions of an equation, if there are any, is often referred to as "solving the equation."

Unfortunately, there is no one method for finding solutions that works for all types of equations. In some cases, such as solutions of quadratic equations, there is simply a formula that gives us the solution or solutions; we will discuss quadratic equations in the next subsection. In other cases, we proceed by assuming that there is a solution, and then manipulate the equation—always making sure that at each step of the way the new version of the equation is still true—until we can, if we are lucky, isolate the unknown.

Here is an example of solving an equation.

Example 8

Solve the equation $8x + 7 = 22 + 5x$.

Solution

We start by bringing all instances of x to the left-hand side of the equation, and all other expressions to the right-hand side of the equation, and then isolate x, which yields

$$\begin{aligned} 8x - 5x &= 22 - 7 \\ 3x &= 15 \\ x &= \frac{15}{3} = 5. \end{aligned}$$

The solution of the equation is therefore $x = 5$.

Observe that solving the equation in Example 8 of this section is an entirely different progress from verifying that $x = 5$ is a solution. If we had been asked to do that verification, we would not have proceed as in the example, but, rather, we would have said the following: "if we substitute $x = 5$ into the left-hand side of the equation we obtain $8x+7 = 8\cdot 5+7 = 47$, and substitute $x = 5$ into the right-hand side of the equation we obtain $22 + 5x = 22 + 5 \cdot 5 = 47$. Because the two sides of the equation are equal when we substituted in $x = 5$, then $x = 5$ is a solution of the equation."

The method of solving the equation in Example 8 of this section worked, and it is what most people would do to solve that equation, but it does have a logical problem that should be noted, because it is a logical problem that must be avoided in more theoretical aspects of mathematics. The logical problem is that what we did was logically backwards, in that we started by assuming that "x" is a solution of the equation, and we then deduced what x must be. This backwards approach in Example 8 of this section did produce the correct solution of our problem, because all steps in that particular calculation are reversible. Not all steps in every calculation are reversible, however, as we now see.

Suppose that we are asked to solve the equation

$$\sqrt{10x + 18} = \sqrt{6 + 7x}.$$

We proceed similarly to Example 8 of this section, except that we first need to square both sides of the equation, which allows us to access the unknown x. We then compute

$$\begin{aligned} \sqrt{10x + 18} &= \sqrt{6 + 7x} \\ (\sqrt{10x + 18})^2 &= (\sqrt{6 + 7x})^2 \\ 10x + 18 &= 6 + 7x \\ 10x - 7x &= 6 - 18 \\ 3x &= -12 \\ x &= \frac{-12}{3} = -4. \end{aligned}$$

✗ Incorrect Solution

The above write-up, though commonly seen, is definitely not correct, because $x = -4$ is in fact not a solution of the original equation. To see that it is not a solution, let us substitute it into each of the two sides of the equation, to see if they are equal. The problem is that if we substitute $x = -4$ into the left-hand side of the equation, we obtain $\sqrt{10x + 18} = \sqrt{10\cdot(-4) + 18} = \sqrt{-40 + 18} = \sqrt{-22}$, but we cannot take the square root of a negative number, and so $x = -4$ is not a possible solution of the equation. (There is, in fact, no solution of this equation in the context of the real numbers.) The source of the problem is, as noted above, that not every step in a calculation is reversible, and in this case it is the step where we took the square of both sides of the equation that cannot be reversed, because we can take the square of any number, but we cannot take the square root of any number.

The above example does not mean that it is wrong in all cases to solve an equation by assuming it is true, and manipulating the equation until we isolate the unknown, but care must be taken to be certain that every step of the process is reversible.

Equations Involving Rational Expressions

One type of equation that occasionally needs to be solved in calculus are equations involving rational expressions. A basic strategy for solving such equations involves cross multiplying.

Here is an example of using cross multiplying to solve an equation.

Example 9

Solve the equation $\frac{x}{x+3} = \frac{2x-5}{x-1}$.

Solution The original equation is

$$\frac{x}{x+3} = \frac{2x-5}{x-1},$$

and we rewrite this equation by cross multiplying, which yields

$$x(x-1) = (2x-5)(x+3).$$

Observe that we must have the parentheses in the above equation. We then multiply out both sides, obtaining

$$x^2 - x = 2x^2 + 6x - 5x - 15,$$

which we then rearrange as

$$x^2 + 2x - 15 = 0.$$

Factoring the left-hand side yields

$$(x+5)(x-3) = 0,$$

and so the solutions of the equation are $x = -5$ and $x = 3$.

Equations that need to be solved in calculus often involve setting some expression to 0, and there is a particularly nice way to handle equations involving a rational expression equal to 0. For a fraction to equal zero, it must be the case that the numerator equals 0, and hence to solve an equation involving a rational expression equal to 0, it suffices to set the numerator equal to 0 and then to solve that equation.

Rational Expressions Equal to Zero

Let $f(x)$ and $g(x)$ be expressions involving the unknown x. To solve the equation

$$\frac{f(x)}{g(x)} = 0,$$

it suffices to solve the equation

$$f(x) = 0.$$

Here is an example of solving an equation with a rational expression equal to 0.

Example 10

Solve the equation $\frac{x^2 - 5x - 6}{e^x + 4\sin x + 10} = 0$.

Solution The original equation is

$$\frac{x^2 - 5x - 6}{e^x + 4\sin x + 10} = 0,$$

and it can be solved by setting the numerator to 0, which yields

$$x^2 - 5x - 6 = 0.$$

Factoring the left-hand side yields

$$(x - 6)(x + 1) = 0,$$

and so the solutions of the equation are $x = 6$ and $x = -1$.

Example 11

Solve the equation $\frac{\frac{x}{2x-7} + \frac{2}{x-9}}{\frac{x-8}{x-3} + \frac{4}{x}} = 0$.

Solution The original equation is

$$\frac{\frac{x}{2x-7} + \frac{2}{x-9}}{\frac{x-8}{x-3} + \frac{4}{x}} = 0,$$

and it can be solved by setting the numerator to 0, which yields

$$\frac{x}{2x - 7} + \frac{2}{x - 9} = 0.$$

We simplify this equation, and we obtain

$$\frac{x(x-9) + 2(2x-7)}{(2x-7)(x-9)} = 0,$$

which in turn yields

$$\frac{x^2 - 9x + 4x - 14}{(2x-7)(x-9)} = 0,$$

and we then obtain

$$\frac{x^2 - 5x - 14}{(2x-7)(x-9)} = 0.$$

This last equation can be solved by setting the numerator to 0, which yields

$$x^2 - 5x - 6 = 0.$$

Factoring the left-hand side yields

$$(x - 7)(x + 2) = 0,$$

and so the solutions of this last equation are $x = 7$ and $x = -2$.

However, before we conclude that we have found the solutions of the original equation, we need to verify whether or not each of the potential solutions $x = 7$ and $x = -2$ would make the denominator of the left-hand side of the original equation equal to 0, which would cause them not to be solutions of the original equation.

We start with $x = 7$, and we substitute that number into the left-hand side of the denominator of the original equation, which yields

$$\frac{7-8}{7-3} + \frac{4}{7} = \frac{-1}{4} + \frac{4}{7} = \frac{(-1)\cdot 7 + 4\cdot 4}{4\cdot 7} = \frac{9}{28} \neq 0,$$

and so the number $x = 7$ is indeed a solution of the original equation.

Next, we try $x = -2$, and we substitute that number into the left-hand side of the denominator of the original equation, which yields

$$\frac{(-2)-8}{(-2)-3} + \frac{4}{-2} = \frac{-10}{-5} + \frac{4}{-2} = 2 + (-2) = 0,$$

and so the number $x = -2$ is not a solution of the original equation.

We conclude that the solution of the original equation is $x = 7$.

☞ **Error Warning** The method of solving an equation in which a rational expression equals 0 by setting the numerator equal to 0 works only for the number 0, and not for any other number.

Solutions of Inequalities

In addition to solving equations in calculus, we also need to solve inequalities. As was the case with equations, so too with inequalities we stress the difference between having a solution of an inequality and finding a solution of an inequality. A solution of an inequality is simply a value (in our case a numerical value) that, when substituted into the inequality, makes the inequality true.

Solution of Inequality

Let $f(x) < g(x)$ or $f(x) \leq g(x)$, respectively, be an inequality with unknown x.

1. A **solution** of this inequality is a number c such that replacing x with c in the inequality, meaning $f(c) < g(c)$ or $f(c) \leq g(c)$, respectively, yields a true statement.

2. An inequality can have one solution, or more than one solution or no solution.

3. In practice, inequalities often have infinitely many solutions, which are written either as inequalities involving the unknown or as intervals.

Here is an example of verifying whether or not a number is a solution of an inequality.

Example 12

Verify whether or not each of the following numbers is a solution of the inequality $x^2 - 3x > x - 3$.

(a) $x = 2$

(b) Any number x in the interval $(3, \infty)$

Solution

(a) If we substitute $x = 2$ into the left-hand side of the inequality, we obtain

$$x^2 - 3x = 2^2 - 3{\cdot}2 = 4 - 6 = -2,$$

and if we substitute $x = 2$ into the right-hand side of the inequality, we obtain

$$x - 3 = 2 - 3 = -1.$$

Because the left-hand side of the inequality is not larger than the right-hand side of the inequality when we substituted in $x = 2$, then $x = 2$ is not a solution of the inequality.

(b) We can rearrange the inequality $x^2 - 3x > x - 3$ as the equivalent inequality $x^2 - 4x + 3 > 0$, and we can then factor the left-hand side of the inequality, yielding $(x - 1)(x - 3) > 0$.

Let x be any number in the interval $(3, \infty)$. Then $x > 3$, and it follows that $x - 1 > 3 - 1 = 2$ and $x - 3 > 3 - 3 = 0$. Because both $x - 1 > 0$ and $x - 3 > 0$, then $(x - 1)(x - 3) > 0$, and it follows that the equivalent inequality $x^2 - 3x > x - 3$ is true. We deduce that the number x, which was an arbitrary choice of a number in the interval $(3, \infty)$, is a solution of the inequality.

What it means to be a solution of an inequality is similar to what it means to be a solution of an equation, though there is a big difference between equations and inequalities in terms of what typical solutions are, in that equations often have one, or a few, solutions, whereas inequalities often have infinitely many solutions, which we conveniently write using interval notation.

Now that we know what it means for a number to be a solution of an inequality, the next question is how we find a solution, and even better find all the solutions, of an inequality, or find that there are no solutions. The process of finding the solution or solutions of an inequality, if there are any, is often referred to as "solving the inequality."

Similarly to the way we often solve equations, which is to manipulate them into equivalent, but simpler, equations, we often solve inequalities by manipulating them into simpler inequalities. We do that by using some basic rules for manipulating inequalities. Just as we are able to manipulate an equation by adding the same expressions to both sides of the equation, or multiplying both sides of the equation by the same expression, we do the same to inequalities, though with one very important caveat, which is that when both sides of an inequality are multiplied (or divided) by a negative number, the direction of the inequality is reversed.

Basic Formulas for Inequalities

Let a, b and c be real numbers.

1. If $a < b$, then $a + c < b + c$.
2. If $a \leq b$, then $a + c \leq b + c$.
3. If $a < b$, and if $c > 0$, then $ac < bc$.
4. If $a \leq b$, and if $c > 0$, then $ac \leq bc$.
5. If $a < b$, and if $c < 0$, then $ac > bc$.
6. If $a \leq b$, and if $c < 0$, then $ac \geq bc$.

Here are four examples of solving inequalities.

Example 13

Solve the inequality $6x + 5 \leq 4x - 11$.

Solution We start by bringing $4x$ to the left-hand side of the inequality, and 5 to the right-hand side of the inequality, which we can do because that is accomplished by adding $-4x$ and -5 to both sides of the inequality, which is valid as noted above. Doing that yields

$$\begin{aligned} 6x - 4x &\leq -11 - 5 \\ 2x &\leq -16 \\ x &\leq \frac{-16}{2} = -8. \end{aligned}$$

The solution of the inequality is therefore all x in the interval $(-\infty, -8]$.

Example 14

Solve the inequality $-x^2 + 20 < 8 - 4x$.

Solution We start by bringing all the expressions to the left-hand side of the inequality, which yields

$$\begin{aligned} -x^2 + 20 + 4x - 8 &< 0 \\ -x^2 + 4x + 12 &< 0 \\ x^2 - 4x - 12 &> 0, \end{aligned}$$

where the last line has the direction of the inequality reversed because it was the result of multiplying both sides of the previous inequality by -1.

Next, we factor the left-hand side of the last inequality, which yields

$$(x - 6)(x + 2) > 0,$$

which is equivalent to the original inequality. The product of two positive numbers or two negative numbers is positive, whereas the product of one positive number and one negative number is negative.

Hence, to solve our inequality, we want to find the values of x for which either $x-6>0$ and $x+2>0$, or $x-6<0$ and $x+2<0$.

The inequalities $x-6>0$ and $x+2>0$ are equivalent to $x>6$ and $x>-2$, and both of those are simultaneously true when $x>6$. Similarly, the inequalities $x-6<0$ and $x+2<0$ are equivalent to $x<6$ and $x<-2$, and both of those are simultaneously true when $x<-2$.

The solution of the inequality is the combined solution from both of these cases, which is all x in the interval $(-\infty,-2)$ or in the interval $(6,\infty)$.

Example 15

Solve the inequality $x^4-3x^3<3x^3-8x^2$.

Solution We start by bringing all the expressions to the left-hand side of the inequality, and then factoring, which yields the equivalent inequality

$$x^4-6x^3+8x^2<0$$
$$x^2(x^2-6x+8)<0$$
$$x^2(x-4)(x-2)<0.$$

We know that if $x=0$, then the left-hand side of the inequality will be 0, which is not less than 0, so $x=0$ is not part of the solution.

Now suppose that $x\neq 0$. Then $x^2>0$, and so the question of whether the whole left-hand side of the inequality is less than 0 depends entirely upon whether or not $(x-4)(x-2)<0$.

The product of two positive numbers or two negative numbers is positive, whereas the product of one positive number and one negative number is negative. Hence, to solve our inequality, we want to find the values of x for which either $x-4>0$ and $x-2<0$, or $x-4<0$ and $x-2>0$.

The inequalities $x-4>0$ and $x-2<0$ are equivalent to $x>4$ and $x<2$, both of which cannot be true simultaneously, and so this case does not contribute to the solution. On the other hand, the inequalities $x-4<0$ and $x-2>0$ are equivalent to $x<4$ and $x>2$, and both of those together can be rewritten as $2<x<4$, which, it is noted, does not contain $x=0$.

The solution of the inequality is therefore all x in the interval $(2,4)$.

Example 16

Solve the inequality $\dfrac{7x}{2x^2+12}>\dfrac{2}{x-3}$.

Solution The natural thing to want to do in this situation would be to cross multiply, which we did in Example 9 of this section in the context of solving an equation. However, we need extra caution when cross multiplying in the context of solving an inequality, because what cross multiplying really consists of is multiplying both sides of the equation or inequality by the denominators of the two sides, and in the case of inequalities if we multiply by a negative number or expression, we need to reverse the direction of the inequality.

In our particular case, the denominator $2x^2 + 12$ on the left-hand side of the inequality is always positive, so there is no problem with it, but the denominator $x - 3$ on the right-hand side of the inequality could be either positive or negative, depending upon the value of x, and we need to take that into account by considering two cases, as follows. Observe that $x - 3 \neq 0$, because it is in the denominator of a rational expression, so we need not consider that case.

First, suppose that $x - 3 > 0$. We can then cross multiply without reversing the direction of the inequality, which yields

$$7x(x - 3) > 2(2x^2 + 12).$$

We then multiply out both sides, obtaining

$$7x^2 - 21x > 4x^2 + 24,$$

which we then rearrange as

$$3x^2 - 21x - 24 > 0,$$

and which we further simplify by dividing both sides by 3, yielding

$$x^2 - 7x - 8 > 0.$$

Factoring the left-hand side yields

$$(x - 8)(x + 1) > 0.$$

Similarly to what we saw in Example 14 of this section, this last inequality has solutions $x < -1$ or $x > 8$. However—though it would be very easy to forget—we are assuming here that $x - 3 > 0$, which is the same as $x > 3$. The only values of x that simultaneously satisfies $x > 3$ and either $x < -1$ or $x > 8$ are those values of x that satisfy $x > 8$.

Next, suppose that $x - 3 < 0$. When we cross multiply the original inequality in this case we do reverse the direction of the inequality, which yields

$$7x(x - 3) < 2(2x^2 + 12).$$

A similar calculation as before, though with the inequality reversed, yields the equivalent inequality

$$(x - 8)(x + 1) < 0.$$

This last inequality has solutions $-1 < x < 8$. However, we are assuming that $x - 3 < 0$, which is the same as $x < 3$. The only values of x that simultaneously satisfies $x < 3$ and $-1 < x < 8$ are those values of x that satisfy $-1 < x < 3$.

The solution of the inequality is the combined solution from both of these cases, which is all x in the interval $(-1, 3)$ or in the interval $(8, \infty)$.

As we see in Example 16 of this section, the issue of whether or not we need to reverse the direction of an inequality when we multiply and divide both sides of the inequality makes solving some inequalities trickier than solving similar equations.

☞ **Error Warning** When multiplying an inequality by a negative number or by an expression that has a negative value, make sure to reverse the direction of the inequality.

Inequalities with Absolute Value

Some of the inequalities encountered in calculus involve absolute value; such inequalities are seen, for example, when finding the interval of convergence of power series.

The key to solving inequalities involving absolute value is to convert such inequalities into equivalent ones that do not involve absolute value, which we do with the following facts.

Basic Inequalities with Absolute Value

Let a and r be real numbers.

1. $|a| < r$ if and only if $-r < a < r$.

2. $|a| \le r$ if and only if $-r \le a \le r$.

3. $|a| > r$ if and only if $a < -r$ or $a > r$.

4. $|a| \ge r$ if and only if $a \le -r$ or $a \ge r$.

Here is an example of solving inequalities involving absolute value.

Example 17

Solve each of the following inequalities.

(a) $|x| < 7$

(b) $|3 - y| \le 5$

(c) $|2w + 4| > 1$

Solution

(a) Using the property $|a| < r$ if and only if $-r < a < r$, we rewrite the inequality $|x| < 7$ as $-7 < x < 7$. The solution of the inequality is therefore all x in the interval $(-7, 7)$.

(b) Using the property $|a| \le r$ if and only if $-r \le a \le r$, we rewrite the inequality $|3 - y| \le 5$ as $-5 \le 3 - y \le 5$, which is equivalent to $-8 \le -y \le 2$, which is then equivalent to $-2 \le y \le 8$. The solution of the inequality is therefore all y in the interval $[-2, 8]$.

(c) Using the property $|a| > r$ if and only if $a < -r$ or $a > r$, we rewrite the inequality $|2w + 4| > 1$ as $2w + 4 < -1$ or $2w + 4 > 1$, which is equivalent to $2w < -5$ or $2w > -3$, which is then equivalent to $w < -\frac{5}{2}$ or $w > -\frac{3}{2}$. The solution of the inequality is therefore all w in the interval $(-\infty, -\frac{5}{2})$ or in the interval $(-\frac{3}{2}, \infty)$.

EXERCISES – Chapter 2

1–8 ■ Multiply and then simplify each expression.

1. $(3x + 5)x^2 - 2x + 4$

2. $3x + 5(x^2 - 2x + 4)$

3. $(3x + 5)(x^2 - 2x + 4)$

4. $(2m + 3n)3m^2 + 5mn - n^2$

5. $2m + 3n(3m^2 + 5mn - n^2)$

6. $(2m + 3n)(3m^2 + 5mn - n^2)$

9–12 ■ Multiply and then simplify each expression by using basic algebra formulas.

9 . $(5a+3)^2$

10 . $(3m-4n)^2$

11 . $(5y+1)(5y-1)$

12 . $(2s^2-t)(2s^2+t)$

13–16 ■ Factor each expression by using basic algebra formulas.

13 . $x^2+8x+16$

14 . $4m^2-12mn+9n^2$

15 . z^2-49

16 . $9y^4-16x^6$

17–20 ■ Simplify each expression

17 . $\dfrac{\frac{3}{x^2}}{\frac{6}{x^5}}$

18 . $\dfrac{\frac{2}{a}+1}{\frac{4}{a^2}-1}$

19 . $\dfrac{\frac{1}{n+1}-\frac{1}{n+2}}{\frac{5}{n+1}-3}$

20 . $\dfrac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$

21–28 ■ Solve each equation.

21 . $x^2+2x-8=0$

22 . $x^2-10x+25=0$

23 . $x^2-9x+20=0$

24 . $x^2-3x-28=0$

25 . $2x^2+7x+3=0$

26 . $6x^2+11x-10=0$

27 . $x^2+5x-3=0$

28 . $3x^2-x-5=0$

29–32 ■ Solve each equation.

29 . $\dfrac{x^2+2x-3}{3x^2+6x+15}=0$

30 . $\dfrac{x^2-x-6}{x^2-4}=0$

31 . $\dfrac{\frac{1}{x^2}-\frac{1}{5^2}}{x-5}=0$

32 . $\dfrac{\frac{5}{x+2}-\frac{3}{x}}{\frac{7}{x+4}-\frac{1}{x}}=0$

33–42 ■ Solve each inequality.

33 . $3x+5\le 7x+10$

34 . $x^2-3x<2x$

35 . $2x^2+7x+30>x^2-3x+6$

36 . $x^3-4x^2>2x^2-x^3$

37 . $\dfrac{2x}{x^2+5}>\dfrac{1}{x-2}$

38 . $\dfrac{7}{x^2+3}>\dfrac{1}{x-1}$

39 . $|5x-2|<1$

40 . $|4-3x|\le 7$

41 . $|3x+1|>5$

42 . $|8-2x|\ge 1$

3 Functions and Graphs

Functions are the main ingredient in calculus. The two main things we do in calculus, namely, derivatives and integrals, are things that are done to functions.

Functions are also a unifying aspect of mathematics. For example, whereas logarithms and trigonometry seem to be very different, what we are interested in here is logarithmic functions and trigonometric functions, and, even though these two types of functions arise from very different considerations, as functions we treat them just as we do any other functions.

A common misconception is to think of functions simply as formulas, for example $f(x) = x^2$. Whereas it is true that many useful functions are given by formulas, there are also useful functions that are not given by single formulas, not to mention functions not given by formulas at all. The most basic idea of a function is that it takes some sort of object as input (in calculus the input is numbers or vectors, though other types of input are used elsewhere), and for each possible input, there is one and only one output.

There are different ways of representing functions, including

1. Verbally,
2. Numerically (by a table of values),
3. Graphically,
4. By formula (some says "algebraically," though that isn't correct).

All these methods of describing a function are equivalent, and it is important to be able to go from one method to the other, for example to go from formula to graph and vice-versa.

Domain of a Function

Every function takes certain things as inputs. For example, the function $f(x)$ defined by the formula $f(x) = x^2$ can take all real numbers as inputs, whereas the function $g(x)$ defined by the formula $g(x) = \ln x$ can take only positive real numbers as inputs.

In general, a functions can take as inputs things other than numbers, for example the function that takes as inputs all the people in the world, and to each person assigns that person's height in inches. In our present context, however, we are considering only functions with real numbers as inputs, and we then have the following concepts.

Domain and Range

Let $f(x)$ be a function with real numbers as inputs.

1. The **domain** of $f(x)$ is the set of all possible real numbers for which the function produces an output.
2. The **range** of $f(x)$ is the set of all outputs of the function, when everything in the domain of $f(x)$ is substituted into the function.

The range of a function can be useful in some contexts, though for our purpose the domain is the much more important concept.

In more advanced mathematics, the very important concept of the domain of a function is slightly more general than we are using here.

There is no definitive method for finding the domain of a function. However, there are a few things to keep in mind. For example, because it is not possible to divide by zero, we exclude anything from the domain that would lead to dividing by zero. Hence, the domain of the function defined by the formula $f(x) = \frac{1}{x-2}$ is the set of all real numbers other than 2.

Other standard considerations when finding the domain of a function is that we cannot take the square root of a negative number (we are considering only real numbers here); we cannot take the logarithm of a negative number or 0; and we cannot take the tangent of $\ldots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$.

Here is an example of finding the domain of a function.

Example 1

Find the domain of each of the following functions.

(a) $f(x) = \dfrac{1}{\sqrt{x-3}} + \dfrac{6}{2x^2 - 50}$

(b) $k(x) = \ln(x^4 - 16)$

Solution

(a) Because we cannot take the square root of a negative number, the only values of x that allow us to compute $\sqrt{x-3}$ are when $x - 3 \geq 0$, which is the same as $x \geq 3$. However, if $x = 3$, then we would have $\sqrt{x-3} = 0$, and that cannot be in the domain of the function, because we cannot divide by 0. Hence we must restrict the values of x to $x > 3$.

Because we cannot divide be zero, then we cannot allow any values of x such that $2x^2 - 50 = 0$. That equation is the same as $2x^2 = 50$, which is $x^2 = 25$, which has solutions $x = 5$ and $x = -5$. These two values of x are not in the domain.

Because we are restricting the values of x to $x > 3$, then $x = -5$ is automatically excluded, and hence the domain is the set of all real numbers x such that $x > 3$ and $x \neq 5$. We can write the domain in terms of intervals as $(3, 5)$ and $(5, \infty)$.

(b) Because we cannot take the logarithm of a negative number or 0, the only values of x that allow us to compute $\ln(x^4 - 16)$ are when $x^4 - 16 > 0$, which is the same as $x^4 > 16$. If we observe that $\pm\sqrt[4]{16} = \pm 2$, we see that we must restrict the values of x to $x < -2$ and $x > 2$. We can write the domain in terms of intervals as $(-\infty, -2)$ and $(2, \infty)$.

Substituting Numbers and Expressions into Functions

The point of a function is that we put things into it, and get something out of it for each thing we put into it. For example, let $f(x)$ be the function defined by the formula $f(x) = x^2$. Clearly, if we put 3 into the function, we get $f(3) = 3^2 = 9$ as the output. It can certainly happen that different inputs produce the same output. For example, we note that $f(-3) = (-3)^2 = 9$ for this function. The crucial thing to observe is that a single input produces a single output.

For example, let $g(x)$ be the function defined by the formula $g(x) = \sqrt{x}$. First, we observe that the domain of $g(x)$ is the set of all non-negative numbers. More importantly, we note that when we write $\sqrt{x}$, we mean only the positive square root of x. For example, we have $g(4) = \sqrt{4} = 2$. It is certainly true that -2 is also a square root of 4, but we cannot say that $g(4)$ is ± 2, because that would give us two outputs for the single input 4. Hence, we use the standard convention that $\sqrt{x}$ always means the positive square root of x. If we want to obtain the negative square root of a number, we would need a different function, namely, the function $h(x)$ defined by the formula $h(x) = -\sqrt{x}$.

For calculus, we need to substitute not only single numbers into functions, but also more complicated expressions into functions, and we need to combine various instances of substituting expressions into function. The following example shows some instances of such substitution.

Example 2

Let $g(x)$ be the function defined by the formula $g(x) = \frac{1}{2x-3}$. Find and simplify each of the following expressions.

(a) $g(x+5)$

(b) $g(x)+5$

(c) $\dfrac{g(x+h)-g(x)}{h}$

Solution

(a) We compute

$$g(x+5) = \frac{1}{2(x+5)-3} = \frac{1}{2x+2\cdot 5-3} = \frac{1}{2x+7}.$$

(b) We compute

$$g(x)+5 = \frac{1}{2x-3} + 5 = \frac{1+5(2x-3)}{2x-3} = \frac{1+5\cdot 2x-5\cdot 3}{2x-3} = \frac{10x-14}{2x-3}.$$

(c) We compute

$$\frac{g(x+h)-g(x)}{h} = \frac{\frac{1}{2(x+h)-3} - \frac{1}{2x-3}}{h} = \frac{\frac{1}{2x+2h-3} - \frac{1}{2x-3}}{h} = \frac{\frac{1\cdot(2x-3)-1\cdot(2x+2h-3)}{(2x+2h-3)(2x-3)}}{h}$$
$$= \frac{\frac{-2h}{(2x+2h-3)(2x-3)}}{\frac{h}{1}} = \frac{-2h}{(2x+2h-3)(2x-3)}\cdot\frac{1}{h} = \frac{-2}{(2x+2h-3)(2x-3)}.$$

We could have multiplied out the denominator, but in this type of problem it is rarely useful to do so.

☞ **Error Warning** The expression $f(x+h)$ is not the same as $f(x)+h$. When we add h to x "inside" the function, that is not the same as adding h to the "outside" of the function.

The Graph of a Function

One way of describing a function is via a formula, for example the function $f(x)$ defined by the formula $f(x) = x^2$. Another way of describing the same function is visually, via its graph.

The Graph of a Function

Let $f(x)$ be a function. The graph of $y = f(x)$ is the subset of the plane consisting of all points (a, b) that satisfy the equation $f(a) = b$.

For example, for the function $f(x)$ defined by the formula $f(x) = x^2$, the point $(3, 9)$ is in the graph of the function, because $f(3) = 3^2 = 9$, but the point $(-2, 5)$ is not in the graph, because $f(-2) = (-2)^2 = 4 \neq 5$.

To find all the points on the graph of a function, the most direct method would be to take every number in the domain of the function and put it into the function to find the output, and then plot all the points obtained in this way. Of course, doing that is not physically possible, because are infinitely many real numbers in the domain. Nonetheless, we can figure out what the graphs of many functions looks like. For example, the graph of the function $f(x)$ defined by the formula $f(x) = x^2$ is seen in Figure 1 of this chapter.

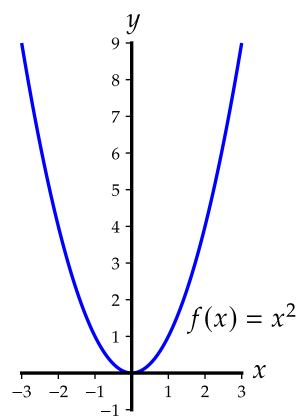


Figure 1: Graph of $f(x) = x^2$

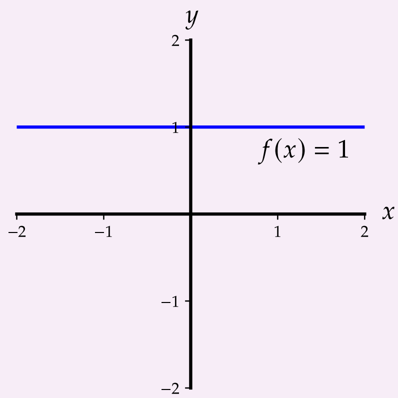
Graphs You Should Know

In the same way that anyone learning a new language needs to know some basic vocabulary and grammar by heart without having to consult a dictionary, anyone who studies calculus should know the graphs of some basic functions—without a calculator—because they are used frequently. These functions will be identified in the various sections of these notes, starting with following.

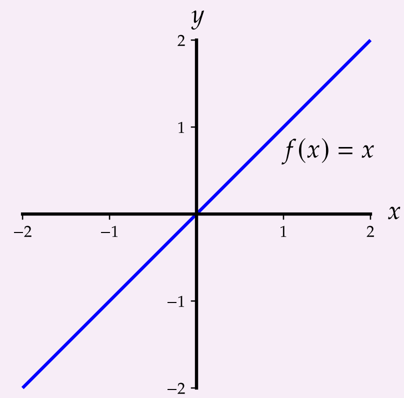
☞ **Need To Know** The graphs of $y = c$, and $y = x$, and $y = -x$ and $y = |x|$.

Basic Graphs

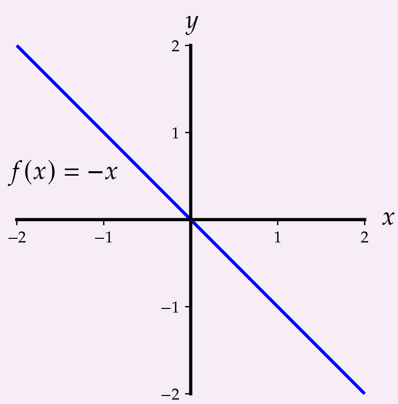
1. $f(x) = c$



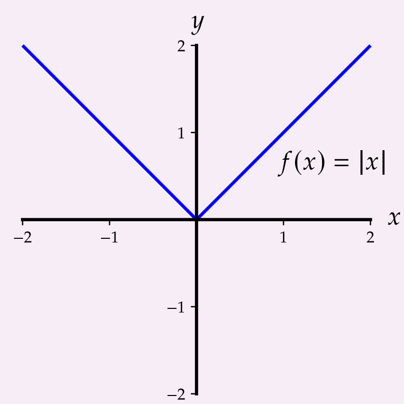
2. $f(x) = x$



3. $f(x) = -x$



4. $f(x) = |x|$



New Graphs from Old

One of the main methods of graphing functions is to do so by modifying the graphs of familiar functions. There are a number of such modifications, which are summarized as follows.

New Graphs from Old

Let $f(x)$ be a function, and let c be a *positive* real number.

Function	Type of Modification
$y = f(x+c)$	shift the graph of $y = f(x)$ to the left by c units
$y = f(x-c)$	shift the graph of $y = f(x)$ to the right by c units
$y = f(x)+c$	shift the graph of $y = f(x)$ upward by c units
$y = f(x)-c$	shift the graph of $y = f(x)$ downward by c units
$y = cf(x)$	stretch the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$	shrink the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$	reflect the graph of $y = f(x)$ in the x-axis
$y = f(-x)$	reflect the graph of $y = f(x)$ in the y-axis

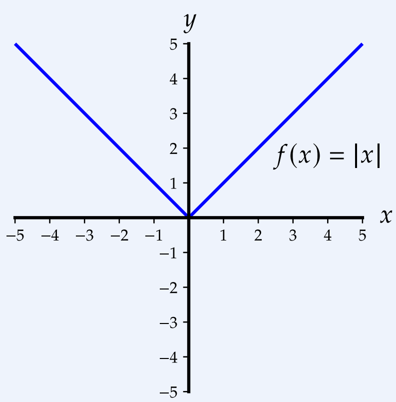
Of course, the various types of modifications listed above can be combined.

Example 3

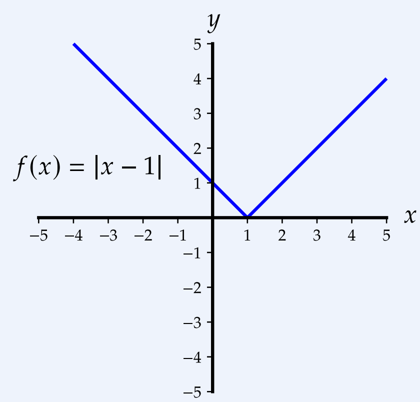
Sketch the graph of the function $f(x) = -2|x-1| + 3$.

Solution As seen in the following figures, we sketch this graph in steps, starting with the graph of $y = |x|$, and then doing one modification of the graph at a time until we obtain the desired result.

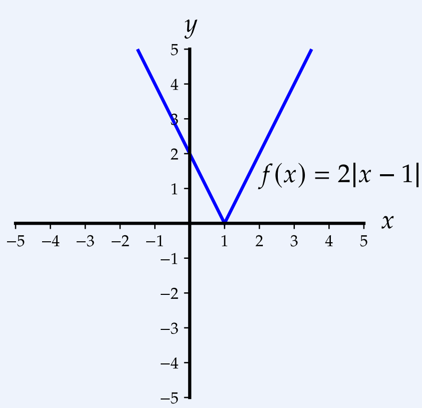
1. $f(x) = |x|$



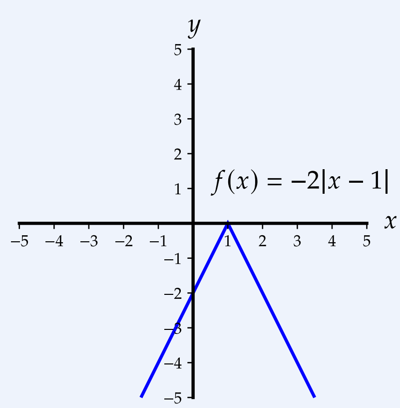
2. $f(x) = |x-1|$



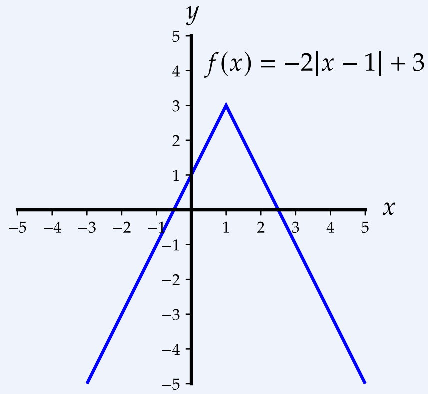
3. $f(x) = 2|x-1|$



4. $f(x) = -2|x-1|$



5. $f(x) = -2|x-1| + 3$



Functions Defined Piecewise

Whereas many commonly used functions are defined by a single formula, for example the function $f(x)$ defined by the formula $f(x) = x^2$, there are many functions that arise in mathematics and its applications that are defined in pieces, rather than by a single formula.

For example, let $g(x)$ be the function defined by the formula

$$g(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ \sin x, & \text{if } x < 0. \end{cases}$$

This function could also be written with interval notation, using the formula

$$g(x) = \begin{cases} x^2, & \text{if } x \text{ is in } [0, \infty) \\ \sin x, & \text{if } x \text{ is in } (-\infty, 0). \end{cases}$$

(In more advanced mathematics texts, the phrase "is in" is denoted by the symbol "$\in$.") This function $g(x)$ is defined for all real numbers x, because the two parts of the function are defined on the intervals $[0, \infty)$ and $(-\infty, 0)$, which together make up all of the real numbers. Moreover, this function is well-defined (that is, its definition makes sense), because there is a unique output for each input, due to the fact that the two parts of the function are defined on intervals that do not overlap.

The above function is and example of a function that is defined piecewise. Of course, a function that is defined piecewise can be defined in more than two parts; any number of parts is acceptable.

The graph of a function that is defined piecewise is plotted by simply graphing each piece of the function on the part of the real numbers for which it is defined. We use a solid dot to indicate the end of a piece of the graph where the graph is actually defined, and a hollow dot to indicate the end of a piece of the graph where the graph is not defined.

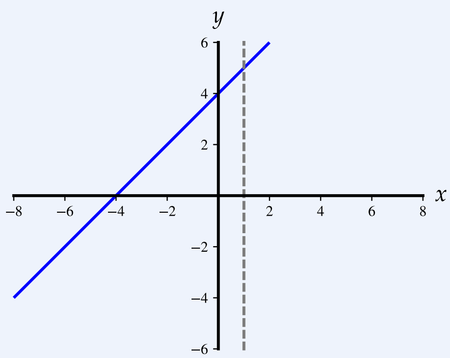
Here is an example of graphing a function that is defined piecewise.

Example 4

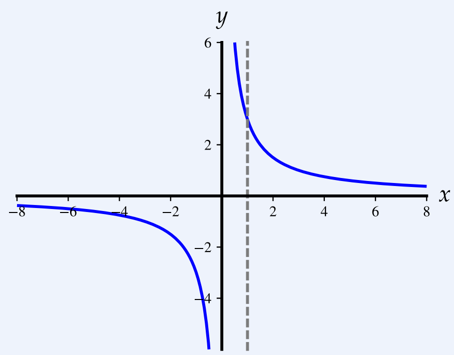
Sketch the graph of the function

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \geq 1 \\ x+1, & \text{if } x < 1. \end{cases}$$

Solution Before drawing the graph of $f(x)$, we see below the graphs of each of the two function used in the pieces of $f(x)$; the dashed vertical line in both of these graphs is $x = 1$.

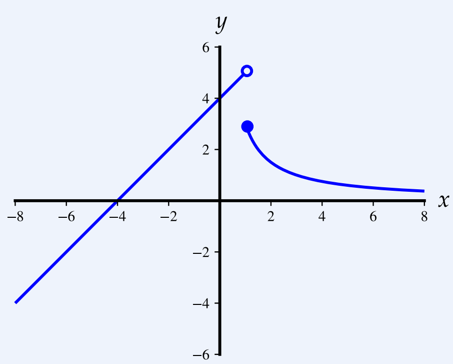


Graph of $g(x) = x + 4$



Graph of $h(x) = \frac{1}{x}$

In the graph of $f(x)$ we use the part of the graph of $y = x + 4$ that is to the left of the dashed line and the part of the graph of $y = \frac{1}{x}$ that is to the right of the dashed line. Putting these two parts together gives the graph of $f(x)$, shown below, where the hollow dot indicates that $y = x + 1$ is define for $x < 1$, which does not include $x = 1$, and the solid dot indicates that $y = \frac{1}{x}$ is define for $x \geq 1$, which does include $x = 1$.



Graph of $f(x)$

How NOT to Graph a Function

There are a number of useful things to do when graphing functions by hand, some of which you will learn in calculus. Without the benefit of calculus, the most basic way to graph functions is to use our familiarity with the graphs of the commonly used functions in precalculus (designated as "graphs you should know" in these notes), together with the method for making new graphs from old, as discussed previously in this section.

By contrast, there is one thing you should definitely not do when you graph a function—which unfortunately is often done in high school—and that is to find the value of the function at a few values of x, to plot those points in the plane and to connect the dots.

As an example of the wrong way to graph a function, suppose we want to plot the graph of $f(x) = x^2$ on the interval $[-3, 3]$. The wrong way to proceed would be to make a chart of some value of $f(x)$, as seen to the left of Figure 2 of this chapter, to plot the corresponding points in the plane, and then to connect the dots, as seen in Figure 2. The result looks roughly, but not exactly, like the actual graph, which is seen in Figure 3 of this chapter.

x	$f(x)$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

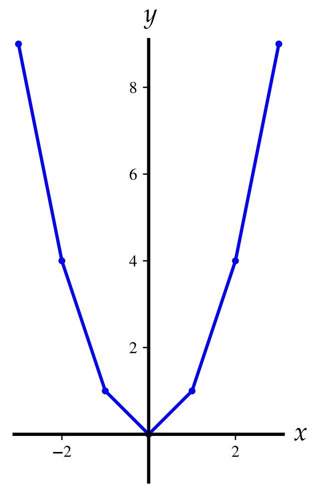


Figure 2: ✗ **Incorrect Graph**

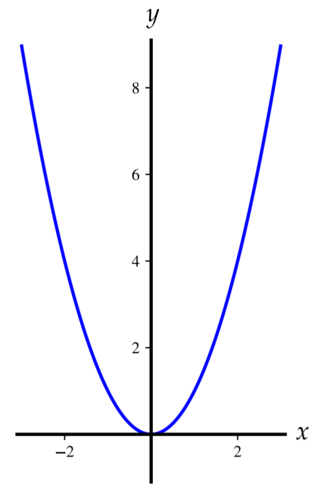


Figure 3: ✓ **Good Graph**

Whereas the above example of the wrong way to plot the graph of a function produces a result that resembles the correct graph (other than the incorrect "corners"), the following example shows that this method of plotting can, in some cases, give very misleading results. This time, suppose we want to plot the graph of $g(x) = \frac{8x-10}{2x-3}$ on the interval $[-2, 5]$. Once again, the wrong way to proceed would be to make a chart of some value of $g(x)$, as seen to the left of Figure 4 of this chapter, to plot the corresponding points in the plane, and then to connect the dots, as seen in Figure 4. The actual graph is seen in Figure 5 of this chapter, and we observe that the result of plotting a few points and connecting the dots completely missed the vertical asymptote. In general, the problem with plotting a few points and connecting the dots is that the chosen few points might miss some important features of the graph.

x	$g(x)$
-2	≈ 3.714
-1	3.6
0	≈ 3.333
1	2
2	6
3	≈ 4.667
4	4.4
5	≈ 4.286

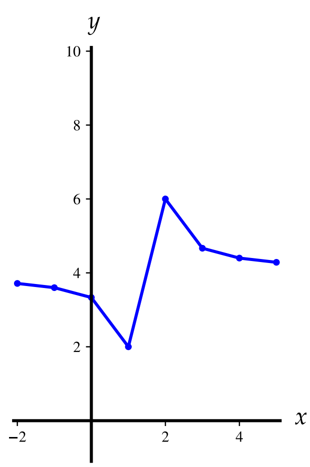


Figure 4: ✗ **Incorrect Graph**

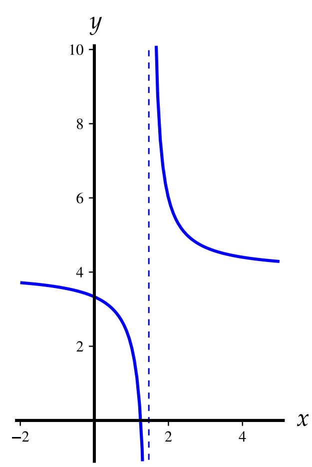


Figure 5: ✓ **Good Graph**

Of course, if we could plot hundreds of very close points by hand, then plotting points and connecting the dots would likely yield a good approximation of the graph—that is precisely how computers plot graphs—but plotting a large number of points by hand is not practical, and plotting just a few points is not a good way to obtain an accurate picture of what the graph actually looks like. Rather, the way to plot a graph is to know what the graph ought to look like, which means to use the basic graphs that you know (lines, quadratics, trigonometric functions, exponentials, logarithms, etc.), and then modify those basic graphs. Once you know some calculus, you can use that to obtain much more information about graphs.

☞ **Error Warning** Do not sketch the graph of a function by plotting a few points and connecting the dots.

Which Curves Are Graphs of Functions?

The graph of every function of the form $y = f(x)$ is a curve in the plane; an example of the graph of a function is seen in Figure 6 of this chapter. On the other hand, not every curve in the plane is the graph of a function; an example of a curve in the plane that is not the graph of a function is seen in Figure 7 of this chapter.

The reason that the curve seen in Figure 7 is not the graph of a function is due to the very essence of the mathematical idea of a function, which is that it takes input from a given set of inputs, and for each input the function gives one and only one output; that idea is true for functions in a variety of mathematical contexts, as the reader will see in more advanced treatments of mathematics, and in particular it is true for functions of the type in which we are interested here, namely, functions that have numbers as input and numbers as output.

Specifically, suppose that $f(x)$ is a function. Then for each number x in the domain of $f(x)$, there is one and only one number y such that $y = f(x)$. Hence, if x is in the domain of $f(x)$, then there is exactly one point on the graph of $y = f(x)$ that corresponds to the number x on the x-axis. If x is a number that is not in the domain of $f(x)$, then there is no point on the graph of $y = f(x)$ that corresponds to the number x on the x-axis. Putting these two cases together, we see that every vertical line in the plane can intersect the graph of $y = f(x)$ either once or not at all; that is, any vertical line intersects the graph in at most one point. Any curve that satisfies this condition with regard to vertical lines is said to pass the Vertical Line Test, and is the graph of a function. By contrast, if a curve in the plane does not satisfy the Vertical Line Test, then it cannot possibly be the graph of a function, that that is what we see in Figure 7.

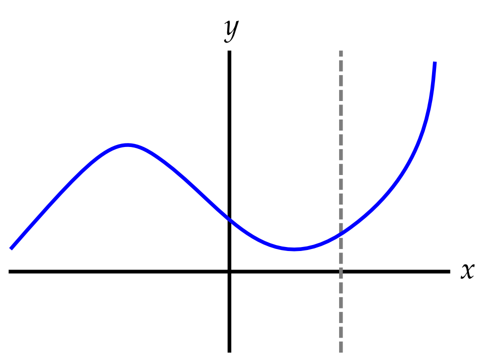


Figure 6: Graph of a function

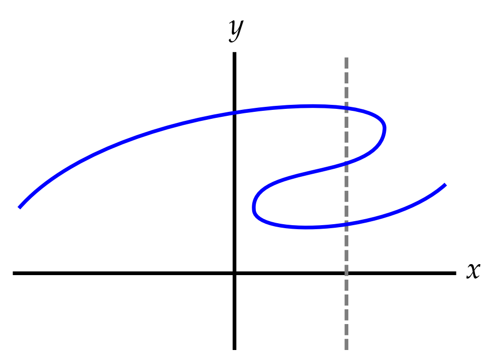


Figure 7: Not the graph of a function

Vertical Line Test

1. A curve in the plane passes the **Vertical Line Test** if every vertical line in the plane intersects the curve in at most one point.

2. A curve in the plane is the graph of a function if and only if the curve passes the Vertical Line Test.

Intervals Where a Function is Greater Than Or Less Than Zero

In Chapter 2 we saw a brief discussion of inequalities, and we now combine inequalities with functions. Specifically, one of the types of inequalities encountered in calculus involves finding when the value of a function is positive or negative, which is used, for example, when finding where a function is increasing or decreasing.

For simple functions, we can find where the function is positive or negative by proceeding just as we did in Chapter 2. For example, suppose we want to find where the function $f(x) = 3x + 5$ is positive; that is, we want to find the values of x for which $f(x) > 0$. To do that, we simply need to solve the inequality $3x + 5 > 0$, which is equivalent to $x > -\frac{5}{3}$, which means that $f(x) > 0$ on the interval $(-\frac{5}{3}, \infty)$. As before, it is convenient, and standard, to give the solution in terms of intervals. A similar calculation shows that $f(x) < 0$ on the interval $(-\infty, -\frac{5}{3})$.

For more complicated functions, however, including many of those that are encountered in calculus, solving an inequality as we just did is often not feasible. Fortunately, there is an alternative method for finding where $f(x)$ is positive or negative that works for continuous functions (and most functions of interest in calculus and the applications of mathematics are continuous); you will learn about continuous functions in a calculus course.

The idea of this method can be seen, for example, in Figure 8 of this chapter, which is the graph of the function $f(x) = \frac{3}{x-2} + 3$. This function is not defined at $x = 2$, and it has a vertical asymptote there. By looking at the graph of this function, we observe that this function is continuous—which means that the function is continuous where it is defined. A common error would be to think that this function is discontinuous at $x = 2$, because the graph is broken up into two pieces at that value of x, but in fact a function can be discontinuous only where it is defined, and because this function is not defined at $x = 2$, it cannot be discontinuous there; it is certainly continuous at all other values of x, which means all values of x where the function is defined.

We are interested in finding where $f(x) > 0$ and where $f(x) < 0$, and for $f(x) = \frac{3}{x-2} + 3$ we see in Figure 8 that $f(x) > 0$ on the intervals $(-\infty, 1)$ and $(2, \infty)$, and that $f(x) < 0$ on the interval $(1, 2)$. Of course, for more complicated functions, we do not always have the graph of the function available to us to see where the function is positive or negative, but what we see in Figure 8 indicates the method we will use even when a graph is not available to us. Specifically, we see in Figure 8 that there are only two places where $f(x)$ changes from positive to negative or vice-versa, and those places are $x = 1$, where the function changes from positive to negative, and $x = 2$, where the function changes from negative to positive. Further, we observe that $f(x) = 0$ at $x = 1$, and $f(x)$ does not exist at $x = 2$.

In general, for a continuous function $f(x)$, the only places where $f(x)$ can change from positive to negative or vice-versa is where $f(x) = 0$ or $f(x)$ does not exist. If, as is often the case, there are only finitely many numbers x where $f(x) = 0$ or $f(x)$ does not exist, then these numbers break up the domain of $f(x)$ into finitely many intervals, and on each of these intervals, either $f(x) > 0$ for all points in the interval or $f(x) < 0$ for all points in the interval. Hence, for each of these intervals, to know whether $f(x) > 0$ for all points in the interval or $f(x) < 0$ for all points in the interval, it suffices to pick any single point in the interval, and check whether $f(x) > 0$ or $f(x) < 0$ at that single point. The number we pick in each interval is arbitrary, and so we typically try to pick a number with which it is easy to work; for example, in the interval

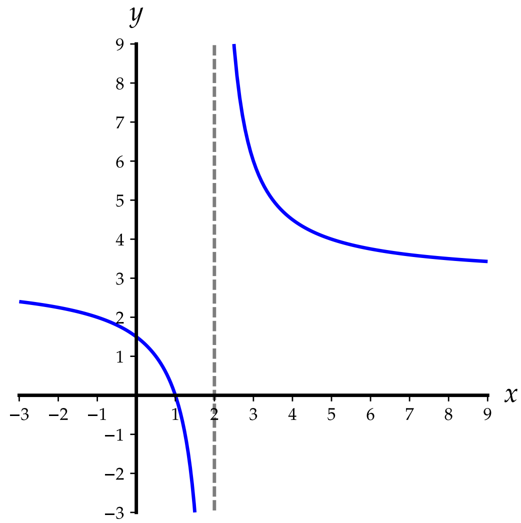


Figure 8: Graph of $f(x) = \frac{3}{x-2} + 3$

$(0, 10)$, we would often pick the test point 1 rather than a number such as 2.73 or π that is less pleasant to use in calculations.

Intervals Where a Function is Positive or Negative

Let $f(x)$ be a function. Suppose that $f(x)$ is continuous, and that the domain of $f(x)$ is an open interval from which at most finitely many points have been removed.

1. The set of points where $f(x) > 0$ consists of finitely many open intervals, and the set of points where $f(x) < 0$ consists of finitely many open intervals.

2. To find the intervals where $f(x) > 0$ and the intervals where $f(x) < 0$, first find the **cutoff points**, which are the numbers x where $f(x) = 0$ or $f(x)$ does not exist. The cutoff points break up the domain of $f(x)$ into open intervals. For each of these intervals, arbitrarily choose a **test point** in the interval, and if $f(x) > 0$ at the test point, then $f(x) > 0$ on the whole interval, and if $f(x) < 0$ at the test point, then $f(x) < 0$ on the whole interval

We note that there are various reasons why a function might not exist at some points, for example to avoid the square root of a negative number, but by far the most common reason in the present context is when the function is in the form of a fraction and the denominator equals 0.

Here is an example of finding where a function is positive or negative. As we see in this example, it is convenient to use a diagram showing the cutoff points and test points. In such diagrams, it is important to mark the test points differently from the cutoff points, to avoid confusion, because the cutoff points and the test points play very different roles.

Example 5

Let $f(x) = x^3 - 10x^2 + 25x$. Find the interval or intervals on which $f(x) > 0$ and the interval or intervals on which $f(x) < 0$.

Solution First, we need to find the cutoff points, which are the numbers x where $f(x) = 0$ or $f(x)$ does not exist. Clearly $f(x)$ exists everywhere. The equation $f(x) = 0$ is $x^3 - 10x^2 + 25x = 0$, which when factored becomes $x(x-5)^2 = 0$, and so this equation has solutions $x = 0$ and $x = 5$; these two numbers are the cutoff points, and they are seen below the x-axis in the following figure.

Cutoff points and test points for $f(x)$

The domain of $f(x)$ is the set of real numbers, and the two cutoff points break up the domain into the three intervals $(-\infty, 0)$, and $(0, 5)$ and $(5, \infty)$.

Second, for each of these three intervals, we choose a test point in the interval; there are infinitely many such choices, but three convenient choices are $x = -1$, and $x = 1$ and $x = 6$, which are seen in the figure above the x-axis and in boxes.

We then calculate

$$\begin{aligned} f(-1) &= (-1)((-1)-5)^2 = -36 < 0, \\ f(1) &= 1 \cdot (1-5)^2 = 16 > 0, \\ f(6) &= 6 \cdot (6-5)^2 = 6 > 0. \end{aligned}$$

We therefore put a minus sign in the figure near $x = -1$, and we put a plus sign in the figure near each of $x = 1$ and $x = 6$. We conclude that

$$\begin{aligned} &f(x) > 0 \text{ on the intervals } (0,5) \text{ and } (5,\infty), \\ &f(x) < 0 \text{ on the interval } (-\infty, 0). \end{aligned}$$

Observe in Example 5 of this section that $f(x) > 0$ on the two consecutive intervals $(0, 5)$ and $(5, \infty)$. There are two common errors associated with this type of situation. First, we note that it is not possible to combine these two intervals into a single interval, because it is not the case that $f(x) > 0$ at $x = 5$. Second, students sometime think that the plus and minus signs in diagrams of the sort we drew in this example must alternative from plus to minus and vice-versa, but, as we saw in the example, that is not a valid assumption.

☞ **Error Warning** It is not necessarily the case that the intervals on which $f(x) > 0$ and the intervals on which $f(x) < 0$ are alternating; a test point must be checked for each interval.

Here is another example of finding where a function is positive or negative.

Example 6

Let $g(x) = \dfrac{x^2 + 3x - 4}{x^2 - 3x}$. Find the interval or intervals on which $g(x) > 0$ and the interval or intervals on which $g(x) < 0$.

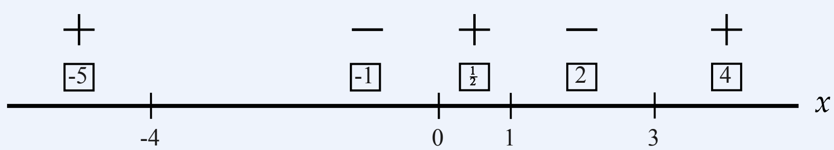
Solution First, we need to find the cutoff points, which are the numbers x where $g(x) = 0$ or $g(x)$ does not exist. To do that, we factor the numerator and denominator of $g(x)$, yielding

$$g(x) = \frac{(x+4)(x-1)}{x(x-3)}.$$

The equation $g(x) = 0$ means that the numerator of $g(x)$ equals 0, which is $(x+4)(x-1) = 0$, which has solutions $x = -4$ and $x = 1$.

The function $g(x)$ does not exist where the denominator of $g(x)$ equals 0, which is $x(x-3) = 0$, which has solutions $x = 0$ and $x = 3$.

The four numbers $x = -4$, and $x = 0$, and $x = 1$ and $x = 3$ are the cutoff points, and they are seen below the x-axis in the following figure.



Cutoff points and test points for $g(x)$

The domain of $g(x)$ is the set of real numbers with the numbers $x = 0$ and $x = 3$ removed, and the four cutoff points break up the domain into the five intervals $(-\infty, 4)$, and $(-4, 0)$, and $(0, 1)$, and $(1, 3)$ and $(3, \infty)$.

Second, for each of these five intervals, we choose a test point in the interval; there are infinitely many such choices, but five convenient choices are $x = -5$, and $x = -1$, and $x = \frac{1}{2}$, and $x = 2$ and $x = 4$, which are seen in the figure above the x-axis and in boxes.

We then calculate

$$\begin{aligned}
g(-5) &= \frac{((-5)+4)((-5)-1)}{(-5)((-5)-3)} = \frac{3}{20} > 0,\\
g(-1) &= \frac{((-1)+4)((-1)-1)}{(-1)((-1)-3)} = -\frac{3}{2} < 0,\\
g(\frac{1}{2}) &= \frac{(\frac{1}{2}+4)(\frac{1}{2}-1)}{\frac{1}{2}(\frac{1}{2}-3)} = \frac{9}{5} > 0,\\
g(2) &= \frac{(2+4)(2-1)}{2(2-3)} = -3 < 0,\\
g(4) &= \frac{(4+4)(4-1)}{4(4-3)} = 6 > 0.
\end{aligned}$$

We therefore put a minus sign in the figure near $x = -1$ and $x = 2$, and we put a plus sign in the figure near each of $x = -5$, and $x = \frac{1}{2}$ and $x = 4$. We conclude that

$$\begin{aligned}
&g(x) > 0 \text{ on the intervals } (-\infty, -4), \text{ and } (0, 1) \text{ and } (3, \infty),\\
&g(x) < 0 \text{ on the interval } (-4, 0) \text{ and } (1, 3).
\end{aligned}$$

We note that in Examples 5 and 6 of this section, and in all such computations, it is not actually necessary to find the precise value of $f(x)$ at the various test points; all we need to know is whether $f(x)$ is positive or negative at the test points, and it is sometimes possible to see whether $f(x)$ is positive or negative when some test point of x is substituted in without actually finding the precise value of $f(x)$. In particular, it is often convenient in such situations to use the factored version of $f(x)$, which is why we did that in Examples 5 and 6. For instance, in Example 6, we saw that $g(2) = \frac{(2+4)(2-1)}{2(2-3)}$, and clearly $2+4$, and $2-1$ and 2 are all positive, and $2-3$ is negative, from which we deduce that $g(2) < 0$ without actually computing the numerical value of $g(2)$, which saves effort.

Finally, we note that a very common error in solving problems such as Example 6 of this section is to consider only the numbers where $f(x) = 0$, but to ignore the numbers where $f(x)$ does not exist; as we see in this example, we would not have obtain the correct solution if we had ignored the numbers where $f(x)$ does not exist.

☞ **Error Warning** The cutoff points are the numbers x where $f(x) = 0$ or $f(x)$ does not exist; do not neglect the numbers x where $f(x)$ does not exist.

EXERCISES – Chapter 3

1–4 ■ Find the domain of each function.

1. $f(x) = \dfrac{1}{x+2} - \dfrac{3}{x^2-9}$ **2.** $h(x) = \sqrt{x^2-16}$

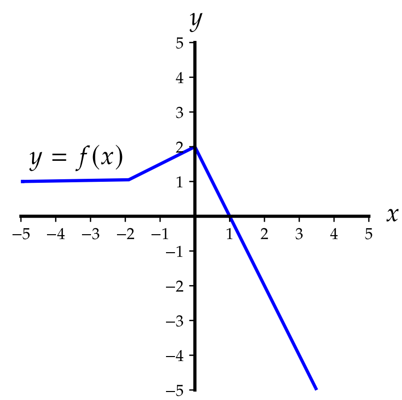
3. $y = \dfrac{1}{\sqrt{x+8}}$ **4.** $g(x) = \ln(x-3)$

5–8 ■ Let $f(x)$ be the function defined by the formula $f(x) = x^2$. Find and simplify each of the following expressions.

5. $f(x-2)$ **6.** $f(x+h) - f(x-h)$

7. $f(x+a+b)$ **8.** $\dfrac{f(x+h) - f(x)}{h}$

9–12 ■ Sketch the graph of each function, where the graph of $y = f(x)$ is seen below.



9. $y = f(x-3)$ **10.** $y = f(x) + 2$

11. $y = -2f(x)$ **12.** $y = 2f(x+1) - 3$

13–16 ■ Sketch the graph of each function.

13. $y = |x+2|$ **14.** $y = |x| + 2$

15. $y = 2|x-3|$ **16.** $y = -|x| + 1$

17–20 ■ Sketch the graph of each function.

17 . $y = \begin{cases} x^2 + 1, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$

18 . $y = \begin{cases} |x - 3|, & \text{if } x \geq 1 \\ 5x^2, & \text{if } x < 1. \end{cases}$

19 . $y = \begin{cases} 3, & \text{if } x \geq 1 \\ x, & \text{if } -1 \leq x < 1 \\ -3, & \text{if } x < -1. \end{cases}$

20 . $y = \begin{cases} \sin x, & \text{if } x \geq \frac{\pi}{2} \\ \tan x, & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \cos x, & \text{if } x \leq -\frac{\pi}{2}. \end{cases}$

21–24 ■ For each function $f(x)$, find the interval or intervals on which $f(x) > 0$ and the interval or intervals on which $f(x) < 0$.

21 . $f(x) = x^3 + 4x^2$

22 . $f(x) = x^3 + 5x^2 - 14x$

23 . $f(x) = \dfrac{x - 4}{x^3 + 3x^2}$

24 . $f(x) = \dfrac{x^2 - 2x}{x^2 + x - 30}$

4 Linear Functions

Linear functions, which are functions the graphs of which are straight lines, appear throughout mathematics and its applications in the sciences and social sciences. In particular, linear functions play a crucial role in calculus, because the derivative of a function is just the slope of the tangent line at each point of the graph of the function.

It is assumed that the reader is familiar with straight lines in the plane from a geometric point of view. It is, however, very important to distinguish between all lines in the plane on the one hand, and lines that are graphs of functions on the other hand. Any line in the plane is given by an equation of the form $ax + cy = d$, whereas a line in the plane that is the graph of a function cannot be vertical, and is given by an equation of the form $y = mx + b$; though we will not be using vertical lines, such a line is given by an equation of the form $x = p$.

Straight lines exist in both the plane and three-dimensional space (and higher-dimensional space as well), but we are concerned here only with lines in the plane.

Linear Functions

Linear functions are convenient to use because they have very simple formulas, as follows.

Linear Functions

1. A **linear function** is a function $f(x)$ that can be defined by a formula of the form $f(x) = mx + b$, where m and b are real numbers.
2. The **slope** of the linear function $f(x) = mx + b$ is the number m.
3. The ***y*-intercept** of the linear function $f(x) = mx + b$ is the number b.

For example, the function $f(x) = 3x + 2$ is a linear function with slope $m = 3$ and y-intercept $b = 2$.

The domain of any linear function is the set of all real numbers.

If the slope of a linear function is not 0, the range of the linear function is the set of all real numbers, and if the slope of a linear function is 0, the range of the linear function is the single number b.

The slope and the y-intercept of a linear function have geometric meanings, as follows.

Slope and y-intercept

Let $f(x) = mx + b$ be a linear function, where m and b are real numbers

1. The slope m of the function $f(x)$ can be computed by the formula

$$m = \frac{y_1 - y_0}{x_1 - x_0}, \tag{1}$$

   where (x_0, y_0) and (x_1, y_1) are any two distinct points in the graph of $f(x)$.

2. The y-intercept b of the function $f(x)$ is the value of y where the line intersects the y-axis.

The important thing to observe about the definition of the slope of a line is that no matter which two points on the line (x_0, y_0) and (x_1, y_1) are chosen, the ratio $m = \frac{y_1 - y_0}{x_1 - x_0}$ will always be the same, which means that the slope of a line is well-defined. That would not be true for any curve other than a straight line.

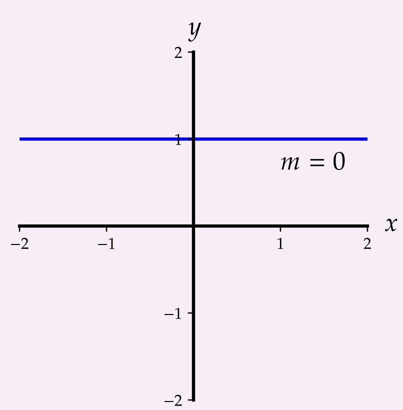
The slope of a line measures how "slanted" the line is. For example, a slope of 0 means that the line is horizontal; a slope of 1 means that the line makes an angle of 45° with the positive x-axis, where positive angles are counterclockwise; and a slope of -1 means that the line makes an angle of $-45°$ with the positive x-axis, where positive angles are clockwise.

There are some basic slopes that you need to know, because they are used frequently.

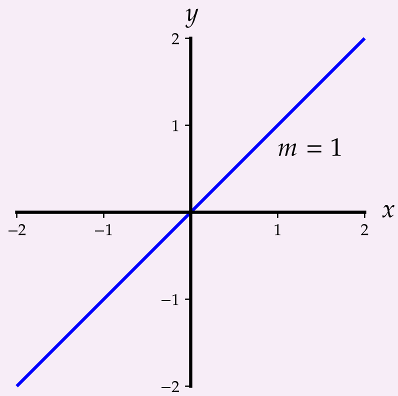
☞ **Need To Know** Slopes $m = 0$, and $m = 1$, and $m = -1$, and $m = 2$ and $m = -2$.

Basic Slopes

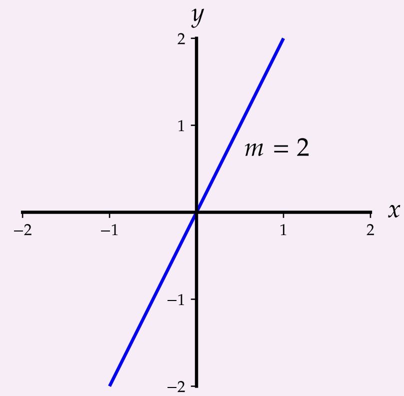
1. $m = 0$



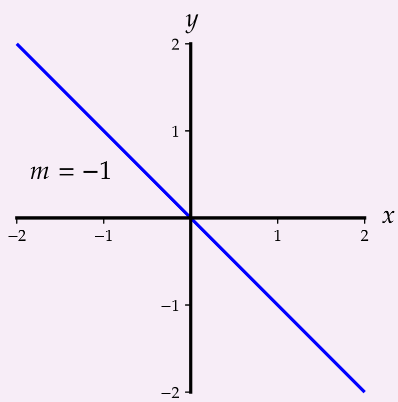
2. $m = 1$



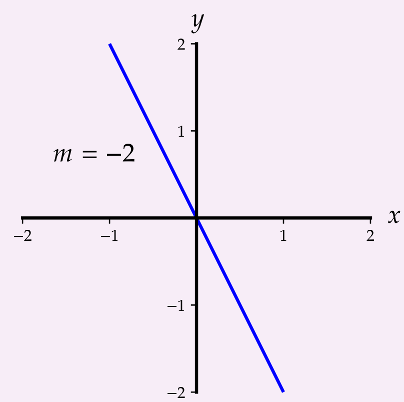
3. $m = 2$



4. $m = -1$



5. $m = -2$



Finding the Equation of a Line

From a geometric perspective, we need two points to determine a line. From an algebraic perspective, we need two pieces of information to determine the linear function representing a line, that is, to determine the slope and y-intercept; these two pieces of information are typically given either as the slope of the line and one point that is contained in the line, or as two points that are contained in the line.

There is more than one method for finding the linear function representing a line given the above type of information, and we use one of these methods here, though other methods work well too.

Finding the Linear Function Representing a Line

1. Let m be a number and let (x_0, y_0) be a point. To find the linear function representing a line with slope m that contains (x_0, y_0), substitute (x_0, y_0) into the equation $y = mx + b$, solve for b, and substitute the result into $y = mx + b$.

2. Let (x_0, y_0) and (x_1, y_1) be points. Suppose $x_0 \neq x_1$. To find the linear function representing a line

that contains (x_0, y_0) and (x_1, y_1), let $m = \dfrac{y_1 - y_0}{x_1 - x_0}$, substitute m into the equation $y = mx + b$, substitute (x_0, y_0) into the equation $y = mx + b$, solve for b, and substitute the result into $y = mx + b$.

Here are two examples of finding the the linear function representing a line, when we are given either a point and a slope, or two points.

Example 1

Find the linear function representing the line containing the point $(3, 2)$ that has slope 4.

Solution A linear function has the form

$$y = mx + b,$$

where m is the slope of the line. We are given that the slope of the line is 4, which means $m = 4$, and so this linear function has the form

$$y = 4x + b.$$

In order to find the value of b, we substitute the point $(3, 2)$ into this equation, where $(3, 2)$ has the form (x, y), and so we substitute $x = 3$ and $y = 2$ into the equation, yielding

$$2 = 4 \cdot 3 + b.$$

Hence

$$b = 2 - 4 \cdot 3 = 2 - 12 = -10.$$

It follows that the linear function representing the line is

$$y = 4x - 10.$$

Example 2

Find the linear function representing the line containing the points $(-4, 1)$ and $(8, 7)$.

Solution A linear function has the form

$$y = mx + b,$$

where m is the slope of the line.

We first need to find the value of m. Using Equation (1) of this chapter with the two points $(-4, 1)$ and $(8, 7)$, we compute

$$m = \frac{7 - 1}{8 - (-4)} = \frac{6}{12} = \frac{1}{2}.$$

Hence this linear function we are seeking has the form

$$y = \frac{1}{2}x + b.$$

In order to find the value of b, we substitute either of the two given points into this equation; we will use the point $(-4, 1)$, though the other point would have worked just as well. The point $(-4, 1)$ has the form (x, y), and so we substitute $x = -4$ and $y = 1$ into the equation, yielding

$$1 = \frac{1}{2}\cdot(-4) + b.$$

Hence

$$b = 1 - \frac{1}{2}\cdot(-4) = 1 + 2 = 3.$$

It follows that the linear function representing the line is

$$y = \frac{1}{2}x + 3.$$

Parallel and Perpendicular Lines

Intuitively, two lines are parallel if they "go in the same direction." For two lines in the plane (though not in higher dimensions), we can also view the lines as parallel precisely if they do not intersect; if the lines do intersect, then they are not parallel. In practice, we can determine if two lines in the plane are parallel, and also if they are perpendicular, by using the slopes of the two lines.

Parallel and Perpendicular Lines

Let $y = mx + b$ and $y = nx + c$ be linear functions representing lines.

1. The two lines are parallel if and only if $m = n$.
2. The two lines are perpendicular if and only if $m = -\frac{1}{n}$.

Here are two examples of finding the linear function representing a line, when we are given a point and either a line that is parallel to the line we want to find or a line that is perpendicular to the line we want to find.

Example 3

Find the the linear function representing the line containing the point $(2, -5)$ that is parallel to the line given by $y = -6x + 19$.

Solution A linear function has the form

$$y = mx + b,$$

where m is the slope of the line.

We first need to find the value of m. The linear function we are trying to find represents a line parallel to the line given by $y = -6x + 17$, and we see that the slope of the latter line is $n = -6$. Because parallel lines have equal slope, we know the slope of the linear function we want to find is $m = -6$. Hence the linear function we are seeking has form

$$y = -6x + b.$$

In order to find the value of b, we substitute the point $(2, -5)$ into this equation, where $(2, -5)$ is a point of the form (x, y), and so we substitute $x = 2$ and $y = -5$ into the equation, yielding

$$-5 = -6{\cdot}2 + b.$$

Hence

$$b = -5 - (-6){\cdot}2 = -5 + 12 = 7.$$

It follows that the linear function representing the line is

$$y = -4x + 7.$$

Example 4

Find the the linear function representing the line containing the point $(9, 11)$ that is perpendicular to the line given by $6x + 10y = 7$.

Solution A linear function has the form

$$y = mx + b,$$

where m is the slope of the line.

We first need to find the value of m. The linear function we are trying to find is perpendicular to the line given by $6x + 10y = 7$, and so we start by finding the slope of the latter line, which we do by solving for y in $6x + 10y = 7$, obtaining $y = -\frac{3}{5}x + \frac{7}{10}$. We deduce that the slope of this line is $n = -\frac{3}{5}$. The slope of the linear function we are seeking is

$$m = -\frac{1}{-\frac{3}{5}} = \frac{5}{3}.$$

Hence the linear function we are seeking has the form

$$y = \frac{5}{3}x + b.$$

In order to find the value of b, we substitute the point $(9, 11)$ into this equation, where $(9, 11)$ is a point of the form (x, y), and so we substitute $x = 9$ and $y = 11$ into the equation, yielding

$$11 = \frac{5}{3}{\cdot}9 + b.$$

Hence

$$b = 11 - \frac{5}{3}{\cdot}9 = 11 - 15 = -4.$$

It follows that the linear function representing the line is

$$y = \frac{5}{3}x - 4.$$

EXERCISES – Chapter 4

1–4 ■ Find the linear function representing each line.

1 . The line containing the point $(1, 2)$ that has slope 3

2 . The line containing the point $(5, 0)$ that has slope -2

3 . The line containing the point $(2, 1)$ that is parallel to the line $y = 5x - 3$

4 . The line containing the point $(3, 2)$ that is perpendicular to the line $y = 2x + 1$

5–8 ■ Find the linear function representing the line containing each pair of points.

5 . $(1, 2)$ and $(3, 8)$

6 . $(2, 1)$ and $(-5, 6)$

7 . $(3, 4)$ and $(-2, 4)$

8 . $(4, 2)$ and $(-1, 2)$

9–12 ■ For each pair of linear functions, state whether they represent lines that are parallel, perpendicular or neither.

9 . $y = 2x + 5$ and $6x - 3y = 4$

10 . $y = 3x + 4$ and $x + 3y - 1 = 0$

11 . $x - 4y + 2 = 0$ and $2x - 6y = 5$

12 . $y = 7$ and $x = 2$

5 Polynomial Functions

Linear functions, which can be written in the form $f(x) = mx + b$ for some real numbers m and b, are the simplest type of broadly useful function, though of course not everything in the world can be described by such functions. The next simplest type of function is polynomial functions. Of course, all linear functions are polynomial functions, though not vice-versa.

Polynomial Functions

Polynomial functions are defined as follows.

Polynomial Functions

A **polynomial function** is a function $f(x)$ that can be defined by a formula of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are real numbers such that $a_n \neq 0$.

Some basic terminology for polynomial functions is as follows.

Polynomial Function Terminology

Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

be a polynomial function, where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are real numbers such that $a_n \neq 0$.

1. The **coefficients** of the polynomial function $f(x)$ are the numbers $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$.
2. The **leading coefficient** of the polynomial function $f(x)$ is the number a_n, which is never zero.
3. The **constant term** of the polynomial function $f(x)$ is the number a_0.
4. The **degree** of the polynomial function $f(x)$ is the number n.

5. A **quadratic function** is a polynomial function of degree 2.

For example, the function $f(x) = 6x^5 - 3x^4 + 5x^2 + 12$ is a polynomial function of degree 5, with leading coefficient 6 and constant term 12. To make this function precisely fit the definition of a polynomial function, we could rewrite it as $f(x) = 6x^5 - 3x^4 + 0x^3 + 5x^2 + 0x + 12$, so that the coefficients $a_5, a_4, a_3, a_2, a_1, a_0$ are the numbers $6, -3, 0, 5, 0, 12$.

Any linear function, which has the form $f(x) = mx + b$, is a polynomial function of degree 1, with leading coefficient m and constant term b. By contrast, polynomial functions of degree greater than 2 are not linear function.

The domain of any polynomial function is the set of all real numbers.

The range of every odd-degree polynomial function is the set of all real numbers (this fact is evident from the graphs of polynomials, though a proof is subtle).

The range of every even-degree polynomial function is not the whole set of real number. More precisely, if the leading coefficient of an even-degree polynomial is positive, then the polynomial has a minimum value, and the the range consists of all real numbers greater than or equal to this minimum value; if the leading coefficient of an even-degree polynomial is negative, then the polynomial has a maximum value, and the the range consists of all real numbers less than or equal to this maximum value.

One of the common things that need to be done with polynomial functions in calculus is to set a polynomial function equal to 0, and then solve the equation.

Polynomial Equation

A **polynomial equation** is an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0,$$

where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are real numbers such that $a_n \neq 0$.

The solutions of a polynomial equation have a special name.

Roots of Polynomial Function

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial function. A **root** of the polynomial function $f(x)$ is a solution of the polynomial equation $f(x) = 0$, that is, a solution of the equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0.$$

We will discuss roots of polynomial functions further, starting with roots of quadratic functions.

Roots of Quadratic Functions

The simplest type of polynomial equation is a linear equation, for example, $3x + 12 = 0$, and it is assumed the reader is familiar with solving such an equation. The next simplest type of polynomial equation is a quadratic equation, to which we now turn.

Quadratic Equation

A **quadratic equation** is an equation of the form $ax^2 + bx + c = 0$, where a, b and c are real numbers such that $a \neq 0$.

Before considering how to solve a quadratic equation, we first ask how many solutions such an equation might have. The key idea is to look at the graph of a quadratic function, that is, a function of the form $f(x) = ax^2 + bx + c$, where a, b and c are real numbers such that $a \neq 0$. The relevant fact is that the graph of any such function is a parabola. In Figure 1 of this chapter we see the graph of $f(x) = x^2 - 4x$, which intersects the x-axis in 2 points; in Figure 2 of this chapter we see the graph of $f(x) = x^2 - 4x + 4$, which intersects the x-axis in 1 point; and in Figure 3 of this chapter we see the graph of $f(x) = x^2 - 4x + 8$, which does not intersect the x-axis. The three parabolas in these figures are facing up, corresponding to the fact that the coefficient of x^2 is positive, but the same three cases are possible for parabolas that are facing down. That observation tells us the possible number of solutions of any quadratic equation.

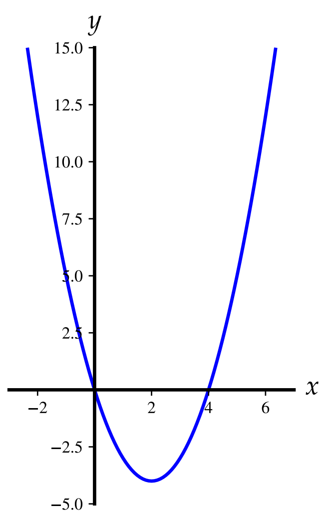


Figure 1: Graph of $f(x) = x^2 - 4x$

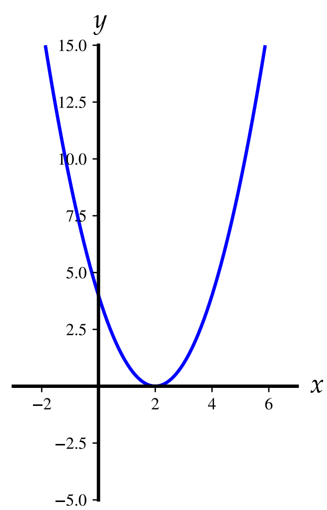


Figure 2: Graph of $f(x) = x^2 - 4x + 4$

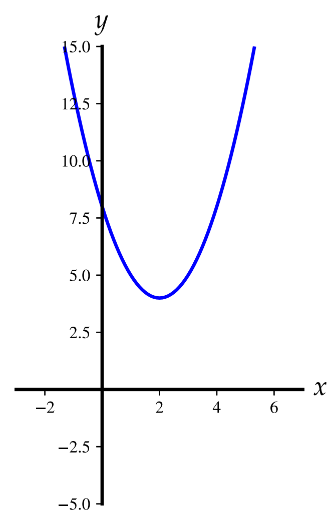


Figure 3: Graph of $f(x) = x^2 - 4x + 8$

Solutions of Quadratic Equations

Let $ax^2 + bx + c = 0$ be a quadratic equation. The quadratic equation has 0, 1 or 2 solutions that are real numbers.

The reader who is familiar with complex numbers might be aware of the fact that in the context of the complex numbers, every quadratic equation has at least one solution. For use in calculus, however, we are mostly interested in solutions of equations that are real numbers, not complex numbers, and so we are focusing here on real numbers only. (That said, there are some instances in calculus, for example solutions of linear differential equations, where complex numbers are in fact needed.)

There are two methods for solving quadratic equations. The quicker method, though it does not work in all cases, is to factor the quadratic. The other method, which works in all cases but is more tedious, is to use the Quadratic Formula.

Solving Quadratic Equations

Let $ax^2 + bx + c = 0$ be a quadratic equation. There are two methods to solve this equation.

1. If numbers r and s can be found such that $r + s = b$ and $rs = c$, then $x^2 + bx + c = (x + r)(x + s)$, and the solutions of $x^2 + bx + c = 0$ are $x = -r$ and $x = -s$.
2. The solutions of $ax^2 + bx + c = 0$ are given by the **Quadratic Formula**, which is
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (1)$$
3. The quadratic equation $ax^2 + bx + c = 0$ has 2 solutions that are real numbers if $b^2 - 4ac > 0$; it has 1 solution that is a real number if $b^2 - 4ac = 0$; and it 0 solutions that are real numbers if $b^2 - 4ac < 0$.

Here are three examples of solving quadratic equations.

Example 1

Solve the equation $x^2 + 5x - 24 = 0$.

Solution We observe that if we multiply the numbers 8 and -3 we obtain $8\cdot(-3) = -24$, and that if we add the numbers 8 and -3 we obtain $8 + (-3) = 5$. Hence the equation $x^2 + 5x - 24 = 0$ can be rewritten as
$$(x + 8)(x - 3) = 0.$$
It follows that
$$x + 8 = 0 \quad \text{or} \quad x - 3 = 0.$$
Hence the solutions of the equation are $x = -8$ and $x = 3$.

Example 2

Solve the equation $x^2 + 3x + 7 = 0$.

Solution There does not appear to be any way to solve this equation by factoring it, so we use the Quadratic Formula, which is Equation (1) of this chapter. Our quadratic equation has $a = 1$, and $b = 3$ and $c = 7$, and we obtain
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4\cdot 1\cdot 7}}{2\cdot 1} = \frac{-3 \pm \sqrt{9 - 28}}{2} = \frac{-3 \pm \sqrt{-19}}{2}.$$
Because the number under the square root is negative, and we cannot take the square root of a negative number in the realm of the real numbers, the equation has no solution in the real numbers.

Example 3

Solve the equation $9x^2 - 42x + 49 = 0$.

Solution We use the Quadratic Formula, which is Equation (1) of this chapter, with $a = 9$, and $b = -42$ and $c = 49$, and we obtain

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-42) \pm \sqrt{(-42)^2 - 4\cdot 9\cdot 49}}{2\cdot 9} = \frac{42 \pm \sqrt{1764 - 1764}}{18} = \frac{42 \pm 0}{18} = \frac{7}{3}.$$

The following example shows a trick that uses quadratic equations to solve a very specific type of quartic (degree 4) polynomial equation, namely, those that have terms with x^4 and x^2, but no other powers of x.

Example 4

Solve the equation $x^4 - 3x^2 - 10 = 0$.

Solution Because the only powers of x in the equation are x^2 and x^4, We can rewrite the equations in terms of powers of x^2 as $(x^2)^2 - 3x^2 - 10 = 0$. We then make the substitution $u = x^2$, and the equation becomes $u^2 - 3u - 10 = 0$, which is a quadratic equation.

We observe that if we multiply the numbers 2 and −5 we obtain $2\cdot(-5) = -10$, and that if we add the numbers 2 and −5 we obtain $2 + (-5) = -3$. Hence the equation $u^2 - 3u - 10 = 0$ can be rewritten as

$$(u + 2)(u - 5) = 0.$$

It follows that

$$u + 2 = 0 \quad \text{or} \quad u - 5 = 0.$$

Hence the solutions of the equation with unknown u are $u = -2$ and $u = 5$.

Using the substitution $u = x^2$, we see that the solutions of the original equation are obtained as the solutions of the two equations $x^2 = -2$ and $x^2 = 5$. The equation $x^2 = -2$ does not have a solution in the real numbers, and the equation $x^2 = 5$ has solutions $x = \sqrt{5}$ and $x = -\sqrt{5}$, which are therefore the solutions of the original equation.

Roots of Polynomial Functions

In the previous subsection we considered the graphs of quadratic functions, which are polynomials of degree 2, and we observed that a quadratic function has either 0, 1 or 2 roots. To see the pattern for higher degree polynomials, we consider, as an example, quartic polynomial functions, that is, polynomial functions of degree 4. In Figure 4 of this chapter we see the graph of $f(x) = x^4 + 2x^3 - 35x^2 - 72x + 64$, which intersects the x-axis in 4 points; in Figure 5 of this chapter we see the graph of $f(x) = x^4 + 2x^3 - 35x^2 - 72x - 36$, which intersects the x-axis in 3 points; in Figure 6 of this chapter we see the graph of $f(x) = x^4 + 2x^3 - 35x^2 - 72x - 236$, which intersects the x-axis in 2 points; in Figure 7 of this chapter we see the graph of $f(x) = x^4 + 2x^3 - 35x^2 - 72x + 464$, which intersects the x-axis in 1 point; and in Figure 8 of this chapter we see the graph of $f(x) = x^4 + 2x^3 - 35x^2 - 72x + 564$, which does not intersect the x-axis. The five quartics in these figures are facing up, corresponding to the fact that the coefficient of x^5 is positive, but the same five cases are possible for quartics that are facing down. These observation suggests that a polynomial function of degree 4 has either 0, 1, 2, 3 or 4 roots, and that leads to a more general pattern for polynomial functions of any degree.

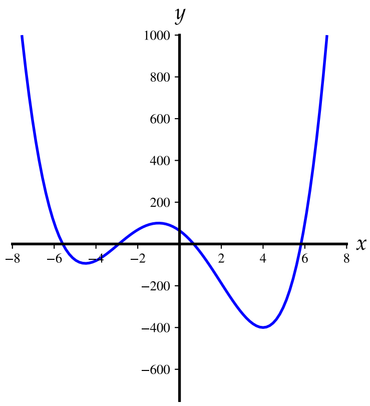


Figure 4: $y = x^4 + 2x^3 - 35x^2 - 72x + 64$

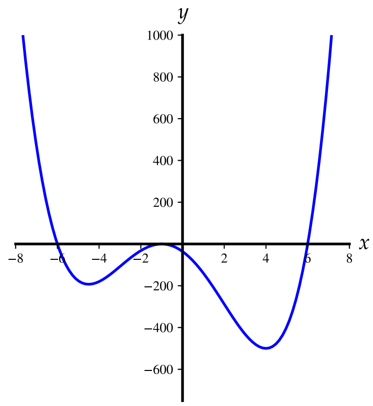


Figure 5: $y = x^4 + 2x^3 - 35x^2 - 72x - 36$

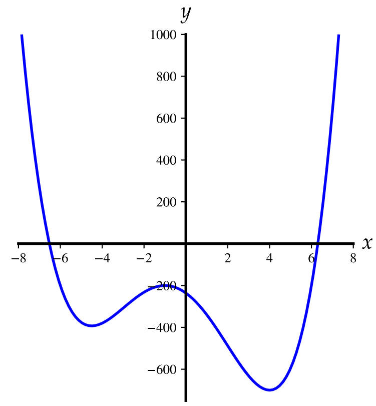


Figure 6: $y = x^4 + 2x^3 - 35x^2 - 72x - 236$

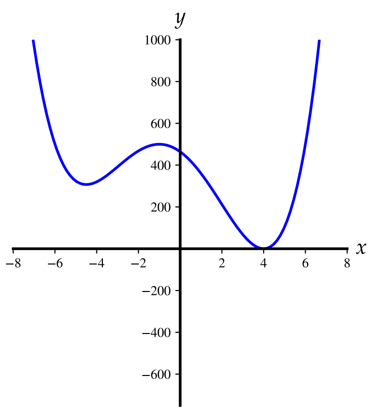


Figure 7: $y = x^4 + 2x^3 - 35x^2 - 72x + 464$

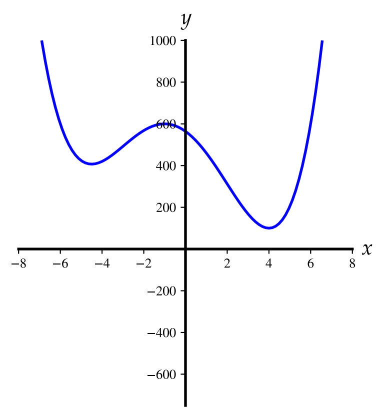


Figure 8: $y = x^4 + 2x^3 - 35x^2 - 72x + 564$

Number of Roots of Polynomial Function

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial function of degree n. The number of roots of the polynomial function $f(x)$ that are real numbers is from 0 to n, inclusive.

The above fact about the number of roots of a polynomial function is correct for all such functions, but it misses one subtlety, which is that the two types of polynomial functions we happened to consider so far when counting the number of possible roots, which are quadratic and quartic polynomial functions, have even degree, and so we have missed an extra fact about odd degee polynomial functions.

To see the pattern for odd degree polynomials, we look, as an example, at cubic polynomial functions, that is, polynomial functions of degree 3. In Figure 9 of this chapter we see the graph of $f(x) = x^3 + 3x^2 - 2$, which intersects the x-axis in 3 points; in Figure 10 of this chapter we see the graph of $f(x) = x^3 + 3x^2 - 4$, which intersects the x-axis in 2 points; and in Figure 11 of this chapter we see the graph of $f(x) = x^3 + 3x^2 - 6$, which intersects the x-axis in 1 point. What we see in these figures is that there is no possible way to move this graph so that it avoids intersecting the x-axis, because as x goes to $-\infty$ the graph goes ever more downward, and as x goes to ∞ the graph goes ever more upwards, and so the graph must intersect the x-axis somewhere. The same observation holds for any polynomial function of odd degree, though if the coefficient of the highest degree term is negative, then upward and downward are reversed. Hence, a polynomial function of odd degree cannot have 0 roots, though it can have any other number of roots up till the degree of the polynomial.

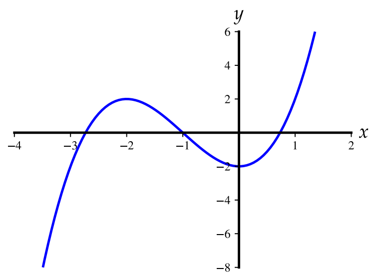


Figure 9: $f(x) = x^3 + 3x^2 - 2$

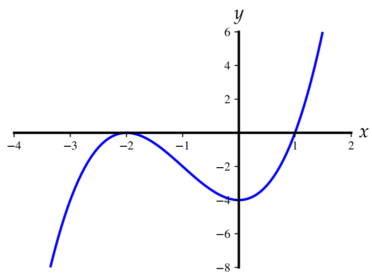


Figure 10: $f(x) = x^3 + 3x^2 - 4$

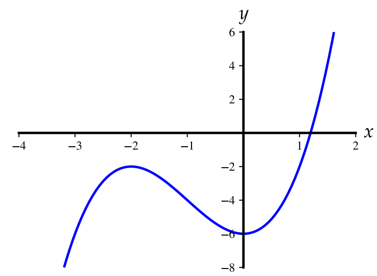


Figure 11: $f(x) = x^3 + 3x^2 - 6$

Number of Roots of Odd Degree Polynomial Function

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial function of degree n, where n is an odd number. The polynomial $f(x)$ has at least one root that is a real number; that is, the number of roots of the polynomial function $f(x)$ that are real numbers is from 1 to n, inclusive.

Having discussed the number of roots a polynomial function can have, we now turn to the question of finding these roots.

It is simple to find the root of a linear polynomial, which means solving an equation of the form $ax+b = 0$. We also know how to find the root of a quadratic polynomial, which means solving an equation of the form $ax^2 + bx + c = 0$, as discussed above.

The obvious question is whether there are methods that allow us to find all the roots of any polynomial function. It turns out that there are formulas for solving polynomial equations of degree 3 and 4, though these formulas are more complicated than the Quadratic Formula. However, rather surprisingly, not only are there no known formulas for solving polynomial equations of degree 5 and above, but it can be proved using more advanced mathematics that no such formulas are possible.

Fortunately, even though we do not have formulas to solve all polynomial equations, we can, with a bit of luck, solve some of these equations. The main method is that in some cases, we can guess one root of the polynomial, and that would allow us to simplify the problem by using the following basic fact about roots of polynomials.

Factoring a Polynomial Function with a Known Root

Let $f(x)$ be a polynomial function of degree n. Let r be a root of $f(x)$. Then $x - r$ is a factor of $f(x)$; that is, there is a polynomial $q(x)$ of degree $n - 1$ such that $f(x) = (x - r)q(x)$.

The benefit of factoring a polynomial function with a known root is as follows. Suppose that we have a polynomial function $f(x)$ of degree n, and that we have a root r of $f(x)$. We can then find a polynomial $q(x)$ of degree $n - 1$ such that $f(x) = (x - r)q(x)$, and in order to find the roots of $f(x)$ other than r, we then try to find the roots of $q(x)$, which might be an easier process, because $q(x)$ has smaller degree than $f(x)$. To find the polynomial $q(x)$, we simply divide $f(x)$ by $x - r$ using long division, which is guaranteed to have remainder 0 because r is a root of $f(x)$. The following example shows this process.

Example 5

Solve the equation $x^3 + x^2 - 17x + 15 = 0$.

Solution Although there is a formula for solving cubic (third degree) equations, it is much more cumbersome to use than the quadratic formula, and in some cases, such as the equation we want to solve, there is an easier method, though it requires a bit of luck. A look at the coefficients of the polynomial $x^3 + x^2 - 17x + 15$ shows that the positive coefficients (which are 1, 1 and 15) add up to 17, and the single negative coefficient is -17. Hence, we guess that $x = 1$ is a root of the polynomial, which we verify by substituting it into the polynomial, which yields $1^3 + 1^2 - 17 \cdot 1 + 15$, which is indeed 0. We have therefore found one root of the polynomial.

Because $x = 1$ is a root of $x^3+x^2-17x+15$, it must be the case that $x-1$ is a factor of $x^3+x^2-17x+15$. We then divide $x^3 + x^2 - 17x + 15$ by $x - 1$ using long division, which is done as follows.

$$
\begin{array}{r|l}
 & \quad\;\; x^2 + \;\; 2x - 15 \\ \hline
x-1 & x^3 + \;\; x^2 - 17x + 15 \\
 & x^3 - \;\; x^2 \\ \cline{2-2}
 & \qquad 2x^2 - 17x \\
 & \qquad 2x^2 - \;\; 2x \\ \cline{2-2}
 & \qquad\quad\; - 15x + 15 \\
 & \qquad\quad\; - 15x + 15 \\ \cline{2-2}
 & \qquad\qquad\qquad 0
\end{array}
$$

As a result of the long division, we see that $x^3 + x^2 - 17x + 15 = (x - 1)(x^2 + 2x - 15)$. We then factor the quadratic polynomial $x^2 + 2x - 15$, which yields $x^2 + 2x - 15 = (x + 5)(x - 3)$, and we deduce $x^3 + x^2 - 17x + 15 = (x - 1)(x + 5)(x - 3)$.

Solving the original equation $x^3 + x^2 - 17x + 15 = 0$ is therefore equivalent to solving $(x - 1)(x + 5)(x - 3) = 0$, and hence the solutions are $x = 1$, and $x = -5$ and $x = 3$.

Multiplicity of Roots of Polynomial Functions

In Example 5 of this section, we started with the polynomial function $f(x) = x^3+x^2-17x+15$, then found that $x = 1$ is a root of $f(x)$, and we were then able to factor the polynomial as $x^3+x^2-17x+15 = (x-1)(x+5)(x-3)$.

For comparison, suppose we had started with the polynomial function $g(x) = x^3 + 2x^2 - 7x + 4$. It can similarly be found that $x = 1$ is also a root $g(x)$, and we can then also factor that polynomial, once again starting out with long division, and at the end of the process (the details of which are left to the reader), it turns $x^3 + 2x^2 - 7x + 4 = (x - 1)(x - 1)(x + 4)$, which can also be written as $(x - 1)^2(x + 4)$.

In both of these examples, we know that because $x = 1$ is a root, then $x - 1$ must be a factor of the polynomial, but what is different between the two examples if that for $f(x)$, the factor $x - 1$ appears only once in the complete factorization $(x - 1)(x + 5)(x - 3)$, whereas for $g(x)$, the factor $x - 1$ appears only once in the complete factorization $(x - 1)^2(x + 4)$. This difference is captured in the following terminology.

Multiplicity of a Root of a Polynomial

Let $f(x)$ be a polynomial function. Let r be a root of $f(x)$. The **multiplicity** of the root r is the largest positive integer m such that $(x-r)^m$ is a factor of $f(x)$.

Here are two examples of finding the multiplicity of the root of a polynomial function, the first of which gives the factorization of the polynomial function, and the second of which does not.

Example 6

Find the multiplicities of the roots of the polynomial function $x^7 - 7x^6 - 6x^5 + 162x^4 - 459x^3 + 405x^2 = (x-3)^4(x+5)x^2$.

Solution By the factorization of the polynomial function, we see that there are three roots, which are $x = 3$, and $x = -5$ and $x = 0$, and that their multiplicities are multiplicity 4 for $x = 3$, and multiplicity 1 for $x = -5$, and multiplicity 2 for $x = 0$.

Example 7

Find the multiplicity of the root $x = 2$ of the polynomial function $x^4 - 5x^3 + 6x^2 + 4x - 8$.

Solution We can verify that $x = 2$ is a root of $x^4 - 5x^3 + 6x^2 + 4x - 8$ by substituting that value of x into the polynomial function, which yields $2^4 - 5 \cdot 2^3 + 6 \cdot 2^2 + 4 \cdot 2 - 8 = 16 - 40 + 24 + 8 - 8 = 0$. Because $x = 2$ is a root of $x^4 - 5x^3 + 6x^2 + 4x - 8$, it must be the case that $x - 2$ is a factor of $x^4 - 5x^3 + 6x^2 + 4x - 8$. We then divide $x^4 - 5x^3 + 6x^2 + 4x - 8$ by $x - 2$ using long division, which is done as follows.

$$
\begin{array}{r|l}
 & x^3 - 3x^2 \qquad + 4 \\
\hline
x - 2 & x^4 - 5x^3 + 6x^2 + 4x - 8 \\
 & \underline{x^4 - 2x^3} \\
 & \quad - 3x^3 + 6x^2 \\
 & \quad \underline{- 3x^3 + 6x^2} \\
 & \qquad\qquad 0 + 4x - 8 \\
 & \qquad\qquad\quad \underline{4x - 8} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

As a result of the long division, we see that $x^4 - 5x^3 + 6x^2 + 4x - 8 = (x-2)(x^3 - 3x^2 + 4)$.

Next, we observe that $x = 2$ is also a root of $x^3 - 3x^2 + 4$, because $2^3 - 3 \cdot 2^2 + 4 = 8 - 12 + 4 = 0$, and hence $x - 2$ is a factor of $x^3 - 3x^2 + 4$. We then divide $x^3 - 3x^2 + 4$ by $x - 2$ using long division, which is done as follows.

$$
\begin{array}{rl}
 & x^2 - x - 2 \\
x - 2 & \overline{\big)\, x^3 - 3x^2 + 4} \\
 & \underline{x^3 - 2x^2} \\
 & - x^2 + 0 \\
 & \underline{- x^2 + 2x} \\
 & - 2x + 4 \\
 & \underline{- 2x + 4} \\
 & 0
\end{array}
$$

It follows that $x^3 - 3x^2 + 4 = (x - 2)(x^2 - x - 2) = (x - 2)(x - 2)(x + 1)$.

We deduce that $x^4 - 5x^3 + 6x^2 + 4x - 8 = (x-2)(x^3 - 3x^2 + 4) = (x-2)(x-2)(x-2)(x+1) = (x-2)^3(x+1)$. We conclude that the root $x = 2$ has multiplicity 3.

The concept of multiplicity of roots of polynomials appears in various parts of mathematics, for example in the solutions of some types of differential equations, which you might see in a calculus course.

Graphs of Polynomial Functions

Let $f(x)$ be a polynomial function of degree n. We saw above that $f(x)$ has anywhere from 0 to n roots, including possibly 0 and n; if n is odd, then $f(x)$ has anywhere from 1 to n roots. It follows that the graph of $f(x)$ intersects the x-axis at most n times. Using calculus, it can further be shown that the graph of $f(x)$ has anywhere from 0 to $n - 1$ "bumps"; if n is even, then the graph of $f(x)$ has anywhere from 1 to $n - 1$ "bumps." (By "bumps" we mean local maxima and local minima, as will be defined in *Calculus I*).

For example, in Figure 4 of this chapter we saw the graph of $f(x) = x^4 + 2x^3 - 35x^2 - 72x + 64$, which has degree 4, intersects the x-axis in 4 points and has 3 bumps; this example has the maximum number of possible intersections with the x-axis and possible bumps. By contrast, in Figure 12 of this chapter we see the graph of $g(x) = x^4 + 2x^3 + 3x^2 - 72x + 264$, which also has degree 4, and which does not intersect the x-axis and has 1 bump; this example has the minimum number of possible intersections with the x-axis and possible bumps.

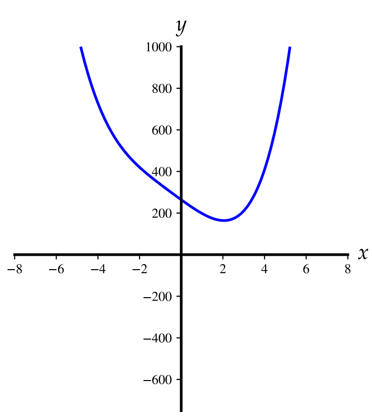


Figure 12: $f(x) = x^4 + 2x^3 + 3x^2 - 72x + 264$

The graphs of polynomial functions do not have asymptotes (unless they are constant polynomials).

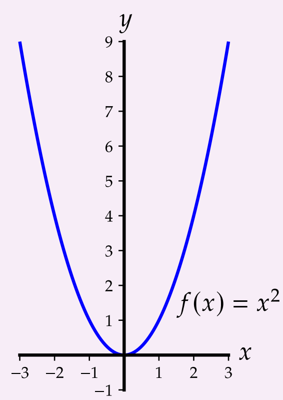
Graphs You Should Know

It is not possible to know the graph of every polynomial, but there are two that are used so often that they are worth knowing.

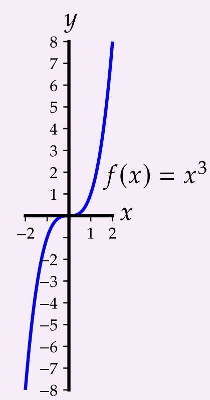
☞ **Need To Know** The graphs of $y = x^2$ and $y = x^3$.

Polynomial Function Graphs

1. $f(x) = x^2$



2. $f(x) = x^3$



The graph of $y = x^3$ appears to be very "thin" in the above figure, because it (and also the graph of $y = x^2$) are shown with aspect ratio 1; that is, the same scale is used for both the x-axis and y-axis. It is important to note that the graph of $y = x^3$ is not at all the same as the central piece of the graph of $y = \tan x$ (seen in Chapter 7), in two ways: the graph of $y = x^3$ does not have any asymptotes, in contrast to the graph of $y = \tan x$, and the graph of $y = x^3$ instantaneously "flattens horizontally" at $x = 0$, which is not true for the graph of $y = \tan x$, where the rigorous idea of "flattening horizontally" uses calculus, and will not be discussed here.

EXERCISES – Chapter 5

1–4 ■ Solve each equation.

1 . $x^3 + 2x^2 - 3x = 0$ **2 .** $x^3 - 2x^2 - x + 2 = 0$

3 . $x^4 - 16 = 0$ **4 .** $x^4 - 7x^2 + 10 = 0$

5–6 ■ For each polynomial function, find the multiplicities of all the roots.

5 . $(x-6)^3(x+3)^5(x-4)$ **6 .** $x^6(x+2)(x-1)^7(x+8)^2$

7–8 ■ For each polynomial function, find the multiplicity of the given root.

7 . $x^3 - 2x^2 - 32x + 96$ and root $x = 4$

8 . $x^3 + 15x^2 + 75x + 125$ and root $x = -5$

9–12 ■ Sketch the graph of each function.

9 . $y = (x-3)^2$ **10 .** $y = x^2 - 5$

11 . $y = 2(x+1)^3$ **12 .** $y = -4x^3 + 1$

6 Power Functions

Polynomials are made up of sums of expressions of the form 1, x, x^2, x^3, x^4, etc., which are multiplied by coefficients. However, while raising x to a positive integer is a particularly simple way to raise x to a power, we can also raise x to numbers that are not positive integers.

Power Functions

The type of function we are now considering is as follows.

Power Functions

1. A **power function** is a function $f(x)$ that can be defined by a formula of the form $f(x) = x^a$, where a is a real number.
2. The **exponent** of the power function $f(x) = x^a$ is the number a.

It is important to stress that the exponent of a power function can be any real number, meaning it can be positive, negative or zero, and it can be an integer, fraction or irrational number. For example, the functions $f(x) = x^{-\frac{2}{3}}$, and $g(x) = x^{\pi}$ and $h(x) = x^{-\sqrt{2}}$ are all power funxtions

Negative and Fractional Powers

We know what a power function is when the exponent is a positive integer. For example, we know that $x^3 = x \cdot x \cdot x$.

Power functions with exponents that are negative integers, or 0 or fractions can be defined as follows.

Power Function Formulas

Let x be a real number, and let n, a and b be positive integers.

1. $x^0 = 1$.
2. $x^{-n} = \dfrac{1}{x^n}$.
3. $x^{\frac{1}{n}} = \sqrt[n]{x}$.
4. $x^{\frac{a}{b}} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$.

Here is an example of converting powers to roots.

Example 1

Evaluate the expression $9^{-\frac{3}{2}}$, without a calculator.

Solution We use the formulas $x^{-n} = \dfrac{1}{x^n}$ and $x^{\frac{a}{b}} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$ to compute

$$9^{-\frac{3}{2}} = \frac{1}{9^{\frac{3}{2}}} = \frac{1}{\left(\sqrt{9}\right)^3} = \frac{1}{3^3} = \frac{1}{27}.$$

Irrational Powers

Whereas defining power functions with exponents that are integers or fractions is straightforward, it is not so simple—though still possible—to define power functions with exponents that are irrational numbers.

Suppose, for example, we want to define a function $f(x)$ by the formula $f(x) = x^{\pi}$. For a function to be meaningful, we need to be able to substitute numerical values for x and be able to obtain the numerical values of the function. But, for this function $f(x)$, what would $f(2)$ mean? That is, how would be compute 2^{π}? The answer is not at all obvious. Certainly, it seems reasonable to assume that because $3 < \pi < 4$, then 2^{π} should be between 2^3 and 2^4, which means that $8 < 2^{\pi} < 16$. That's true, but not satisfactory.

A completely rigorous definition of 2^{π} requires more advanced mathematics than we have at our disposal here, but one approach is as follows. The number π is an irrational number, with decimal expansion that starts $3.14159\ldots$. Hence, we can approximate the value of π by the numbers 3, 3.1, 3.14, 3.141, etc. We can rewrite each of these numbers as fractions, which are $\frac{3}{1}$, $\frac{31}{10}$, $\frac{314}{100}$, $\frac{3141}{1000}$, etc. We know how to raise 2 to each of these exponents, which yields

$$\begin{aligned}
2^3 &= 8 \\
2^{3.1} &= 2^{\frac{31}{10}} = \sqrt[10]{2^{31}} = 8.5741\ldots \\
2^{3.1} &= 2^{\frac{314}{100}} = \sqrt[100]{2^{314}} = 8.8152\ldots \\
2^{3.1} &= 2^{\frac{3141}{1000}} = \sqrt[1000]{2^{3141}} = 8.8213\ldots \\
&\vdots \quad .
\end{aligned}$$

If we keep doing this process, using more and more decimals of π, we will see that the output gets closer and closer to the number $8.8249\ldots$. Then we define $2^{\pi} = 8.8249\ldots$.

The above process makes use of the notion of a limit, which will be discussed in calculus. This process also seems arbitrary, in that the number π can be approached by different sequences of fractions, though it can be proved rigorously that this method always yields the same number, no matter what sequence of fractions is used. A similar approach can be used to find the value of any number raised to an irrational exponent, and hence power functions with exponents that are irrational numbers are indeed defined.

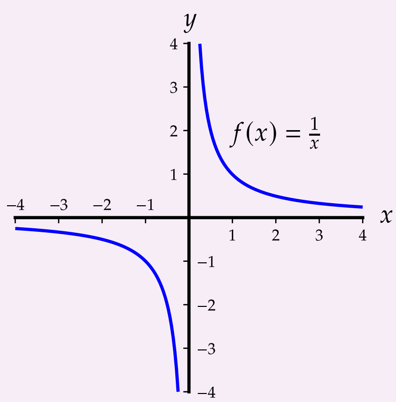
Graphs You Should Know

It is not possible to know the graph of every power function, but there are two that are used so often that they are worth knowing.

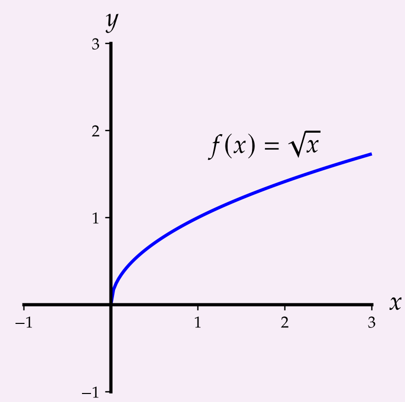
☞ **Need To Know** The graphs of $y = \frac{1}{x}$ and $y = \sqrt{x}$.

Power Function Graphs

1. $f(x) = \dfrac{1}{x}$



2. $f(x) = \sqrt{x}$



Properties of Power Functions

The formulas for defining power functions with exponents that are integers or fractions might seem unmotivated when first encountered, but they make the various properties of power functions, including the following, work out particularly nicely.

Properties of Power Functions

Let x and y be positive real numbers, and let a and b be real numbers.

1. $x^{a+b} = x^a x^b$.

2. $x^{a-b} = \frac{x^a}{x^b}$.

3. $(x^a)^b = x^{ab}$.

4. $(xy)^a = x^a y^a$.

5. $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$.

The last two properties of power functions can be rewritten using roots when the exponent is a fraction of the form $\frac{1}{a}$.

Properties of Roots

Let x and y be positive real numbers, and let a be a real number.

1. $\sqrt[a]{xy} = \sqrt[a]{x}\sqrt[a]{y}$.
2. $\sqrt[a]{\frac{x}{y}} = \frac{\sqrt[a]{x}}{\sqrt[a]{y}}$.

Here are three examples of simplifying expressions using properties of power functions and roots.

Example 2

Simplify the expression $\dfrac{(2x^4y)^3}{10x^7y^{-2}}$ so that it contains only one occurrence of each letter, and has only positive exponents.

Solution We use the formulas $(xy)^a = x^a y^a$, and $(a^x)^y = a^{xy}$ and $x^{a-b} = \frac{x^a}{x^b}$ to compute

$$\frac{(2x^4y)^3}{10x^7y^{-6}} = \frac{2^3(x^4)^3y^3}{10x^7y^{-6}} = \frac{8x^{12}y^3}{10x^7y^{-6}} = \frac{8}{10}x^{12-7}y^{3-6} = \frac{4}{5}x^5y^{-3} = \frac{4x^5}{5y^3}.$$

Example 3

Rewrite the expression $\dfrac{\sqrt{x}}{x^{-4}}$ as a single power.

Solution We use the formulas $x^{\frac{1}{n}} = \sqrt[n]{x}$ and $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$ to compute

$$\frac{\sqrt{x}}{x^{-4}} = \frac{x^{\frac{1}{2}}}{x^{-4}} = x^{\frac{1}{2}-(-4)} = x^{\frac{9}{2}}.$$

Example 4

Rewrite the expression $a^{\frac{2}{3}}b^{-2}$ using positive integer powers, and using a single fraction and/or single root.

Solution We use the formulas $x^{\frac{a}{b}} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$ and $\sqrt[a]{\frac{x}{y}} = \frac{\sqrt[a]{x}}{\sqrt[a]{y}}$ to compute

$$a^{\frac{2}{3}}b^{-2} = \frac{\sqrt[3]{a^2}}{b^2} = \frac{\sqrt[3]{a^2}}{\sqrt[3]{b^6}} = \sqrt[3]{\frac{a^2}{b^6}}.$$

EXERCISES – Chapter 6

1–4 ■ Evaluate each expression, without a calculator.

1 . $4^{\frac{3}{2}}$ **2 .** $27^{\frac{2}{3}}$

3 . $16^{-\frac{1}{4}}$ **4 .** $1000^{-\frac{4}{3}}$

5–8 ■ Rewrite each expression as a single power.

5 . $\sqrt{a^5}$ **6 .** $\dfrac{1}{\sqrt[3]{x^2}}$

7 . $x^{-3}\sqrt{x}$ **8 .** $\dfrac{y^2}{\sqrt[4]{y}}$

9–12 ■ Rewrite each expression using positive powers, and a single fraction and/or single root.

9 . x^{-3} **10 .** $b^{\frac{2}{5}}$

11 . $y^{\frac{3}{2}}z^{-2}$ **12 .** $\dfrac{a^{-4}}{b^{\frac{1}{3}}}$

13–16 ■ Simplify the following expressions so that each answer contains only one occurrence of each letter, and should have only positive exponents.

13 . $3x^{-2}y^5 \cdot (2x^2y)^3$ **14 .** $(ab^2c^3)^{-3} \cdot (2a^{-3}bc^2)^2$

15 . $\dfrac{3s^4t^{-5}}{6s^{-2}t^{-4}}$ **16 .** $\left(\dfrac{m^{-2}n^5p^2}{m^4n^3p^{-2}}\right)^3$

17–20 ■ Sketch the graph of each function.

17 . $y = \dfrac{1}{x+3}$ **18 .** $y = -\dfrac{2}{x}$

19 . $y = \sqrt{x-1} + 2$ **20 .** $y = -3\sqrt{x} - 1$

7 Trigonometric Functions

When students first encounter trigonometry, it is usually in the context of the study of triangles. Whereas triangles are very important in many parts of mathematics and its applications, for calculus our main interest in trigonometry is not the study of triangles, but is rather the six trigonometric functions, which arise from the study of triangles, but which are also useful in many other context, for example oscillatory motion.

Radians and Degrees

The study of triangles involves the measurement of angles. As with other types of measurements, for example length, volume and weight, the measurement of angles involves units of measurement. And, just as there are various units that are used for the measurement of length (for examples, inches and centimeters), so too for angles there are various units that can be used. The units for measuring angles that is the most commonly used in elementary school, middle school and high school are degrees. For calculus, however, degrees are definitely the wrong units to be used for measuring angles; rather, the only units for measuring angles that should be used for calculus are radians.

The problem with degrees is that they are a completely arbitrary unit of measurement. One degree is obtained by taking a complete angle around a point and dividing it into 360 equal parts. The choice of 360 is completely arbitrary, and any other number could have been used. By contrast, radians involve no arbitrary choice in their definition, other than the use of circles of radius 1.

Specifically, suppose that we are given an angle in the plane, which we will denote by t. We then draw a unit circle (that is, a circle with radius 1) with center at the apex of the angle; if the apex of the angle has coordinates (a, b), then this circle has equation $(x - a)^2 + (y - b)^2 = 1$. The angle t then subtends an arc of the circle, denoted S, as seen in See Figure 1 of this chapter. The radian measure of t is simply the length of the arc S.

Radians

Let t be an angle. The **radian measure** of t is defined to be the length of the arc subtended by the angle in a unit circle (that is, a circle with radius 1) with center at the apex of the angle.

Because we all learn about degrees before learning about radians, an important aspect of using radians is to know how to convert degrees to radians and vice-versa. The key to that conversion is to recall that the circumference of the unit circle is 2π, and the unit circle corresponds to a complete angle around a

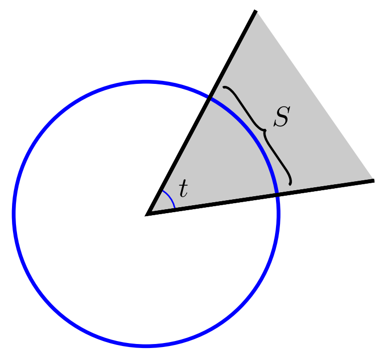


Figure 1: Radians

point, which is 360°. Hence 360° is equal to 2π radians, which leads to the following method for conversion between degrees and radians.

Degrees to Radians Conversion

1. To convert an angle in degrees to radians, multiply by $\dfrac{\pi}{180}$.
2. To convert an angle in radians to degrees, multiply by $\dfrac{180}{\pi}$.

Some regularly used conversions between degrees and radians are the following.

Degrees to Radians Conversion: Standard Angles

1. $90^\circ = \frac{\pi}{2}$ rad.
2. $180^\circ = \pi$ rad.
3. $270^\circ = \frac{3\pi}{2}$ rad.
4. $360^\circ = 2\pi$ rad.

Here are example of converting degrees to radians and vice-versa.

Example 1

Convert the angle 210° to radians, without a calculator.

Solution We know that $180^\circ = \pi$ rad. Hence

$$1^\circ = \frac{\pi}{180}\text{ rad}\,.$$

We multiply both sides of the above equation by 210 to obtain

$$210^\circ = 220\cdot 1^\circ = 220\cdot\frac{\pi}{180}\text{ rad} = \frac{220\pi}{180}\text{ rad} = \frac{11\pi}{9}\text{ rad}\,.$$

Example 2

Convert the angle $-\frac{7\pi}{10}$ rad to degrees, without a calculator.

Solution We know that $180° = \pi$ rad. Hence

$$1 \text{ rad} = \frac{180^\circ}{\pi}.$$

We multiply both sides of the above equation by $-\frac{7\pi}{10}$ to obtain

$$-\frac{7\pi}{10} \text{ rad} = -\frac{7\pi}{10} \cdot 1 \text{ rad} = -\frac{7\pi}{10} \cdot \frac{180^\circ}{\pi} = -\frac{7\pi \cdot 180^\circ}{10\pi} = -126^\circ.$$

The Six Trigonometric Functions

When you first learn about the six trigonometric functions, it is usually in the context of angles in right triangles. That approach is certainly important, but it is also limited in use, because the angles in a right triangle must be between 0° and 90°, or, stated properly in radians, between 0 and $\frac{\pi}{2}$. For calculus and its applications, by contrast, we need the six trigonometric functions defined for all real numbers (in the case of sine and cosine), and almost all real numbers (in the case of tangent, cotangent, secant and cosecant).

In order to define the six trigonometric functions for all (or almost all) real numbers, we use the unit circle, which has equation $x^2 + y^2 = 1$. Specifically, let t be a real number, which we think of as an angle measured in radians. We plot this angle at the origin, starting with the positive x-axis and going counterclockwise, which then gives rise to a ray starting at the origin. This ray intersects the unit circle at a point (x, y). We then form a right triangle, with one side of the triangle the line segment from (x, y) to $(x, 0)$, with the other side of the triangle the line segment from $(x, 0)$ to $(0, 0)$, and with the hypotenuse of the triangle being the radius of the unit circle from $(0, 0)$ to (x, y). See Figure 2 of this chapter for what happens when (x, y) is in the first quadrant; similar figures occur when (x, y) is in the other quadrants.

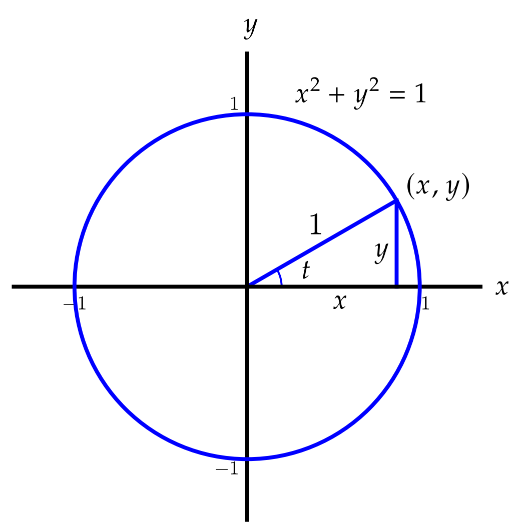


Figure 2: Unit Circle for Trigonometric Functions

We now use this point (x, y) to define the six standard trigonometric functions. (We use the term "standard" here because there are other trigonometric functions that are less widely used, and are not important to know.)

Trigonometric Functions

The six standard **trigonometric functions** are defined as follows. Let t be a real number. Plot an angle measuring t radians with apex at the origin, starting with the positive x-axis and going counterclockwise, which gives rise to a ray starting at the origin. Let (x, y) be the intersection of this ray with the unit circle $x^2 + y^2 = 1$.

1. The **sine** of t is defined by $\sin t = y$.
2. The **cosine** of t is defined by $\cos t = x$.
3. The **tangent** of t is defined by $\tan t = \frac{y}{x}$, except that $\tan t$ is not defined if t is any of $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$.
4. The **secant** of t is defined by $\sec t = \frac{1}{x}$, except that $\sec t$ is not defined if t is any of $\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$.
5. The **cosecant** of t is defined by $\csc t = \frac{1}{y}$, except that $\csc t$ is not defined if t is any of $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$.
6. The **cotangent** of t is defined by $\cot t = \frac{x}{y}$, except that $\cot t$ is not defined if t is any of $\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$. An alternative notation for the cotangent of t is $\operatorname{ctg} t$.

If t is between 0 and $\frac{\pi}{2}$, then the above definition of the six trigonometric functions of t is the same as the definition given for angles in a right triangle. For example, we observe that $\tan t$ in a right triangle is "opposite over adjacent," and because the side opposite the angle t has length y, and the side adjacent to the angle t has length x, then the above definition of $\tan t$ is identical to opposite over adjacent. Similar observations work for the other trigonometric functions, making use of the fact that for the right triangle drawn in the unit circle, the hypotenuse has length 1.

Trigonometric Functions of Standard Angles

For most angles, a calculator or computer is needed to calculate the various trigonometric functions of that angle. However, there a few angle that occur so frequently that it is worth knowing the values of sine and cosine of these angle. These values are as follows.

Sine of Standard Angles

Radians	$\sin 0$	$\sin \frac{\pi}{6}$	$\sin \frac{\pi}{4}$	$\sin \frac{\pi}{3}$	$\sin \frac{\pi}{2}$
Degrees	$\sin 0°$	$\sin 30°$	$\sin 45°$	$\sin 60°$	$\sin 90°$
Value	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Cosine of Standard Angles

Radians	$\cos 0$	$\cos \frac{\pi}{6}$	$\cos \frac{\pi}{4}$	$\cos \frac{\pi}{3}$	$\cos \frac{\pi}{2}$
Degrees	$\cos 0°$	$\cos 30°$	$\cos 45°$	$\cos 60°$	$\cos 90°$
Value	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

The values listed in the chart for cosine are in backwards order from the values listed in the chart for sine; that symmetry makes these values easier to remember. A simple way of remembering these five values is that for sine, the five values in order can be rewritten as

$$\frac{\sqrt{0}}{2} \quad \frac{\sqrt{1}}{2} \quad \frac{\sqrt{2}}{2} \quad \frac{\sqrt{3}}{2} \quad \frac{\sqrt{4}}{2},$$

which has a conveniently simple pattern.

Using a Calculator for Trigonometric Functions

For any angle other than the standard ones mentioned above, we definitely need a calculator or computer to calculate the various trigonometric functions of that angle. Because degrees are the most commonly used measurement of angles in elementary and secondary schools, the default setting on most calculators is to use angles in degrees. For example, using the default settings for most calculators, if you enter the number $\pi \div 2 \approx 1.5707963267$, and you then press the `sine` button, the output will be 0.0274121335, which is $\sin \frac{\pi}{2}°$, and that is not what we mean when we write "$\sin \frac{\pi}{2}$" in calculus.

For calculus, by contrast, we always need to think of the input of the trigonometric functions as radians. To use a calculator with radians, you first need to change the setting on the calculator from degrees to radians. The way to change that setting varies from one calculator to another; some calculators have a button that says `DRG`—which stands for degrees, radians and gradians (the last of these is not important)—and which should be pressed repeatedly until it shows that radians are being used, and other calculators have a button that says `Rad` or the like. Once a calculator is set to use radians, if you enter the number $\pi \div 2 \approx 1.5707963267$, and you then press the `sine` button, the output will be 1, which is $\sin \frac{\pi}{2}$ rad, and that is what we want. Because we always use radians in calculus, we do not normally write "rad" in our calculations, and we did that here just for emphasis; we would normally write "$\sin \frac{\pi}{2}$," where it would be understood that we mean radians.

☞ **Error Warning** When you are using a calculator for evaluating trigonometric functions, make sure the calculator is set for radians rather than degrees.

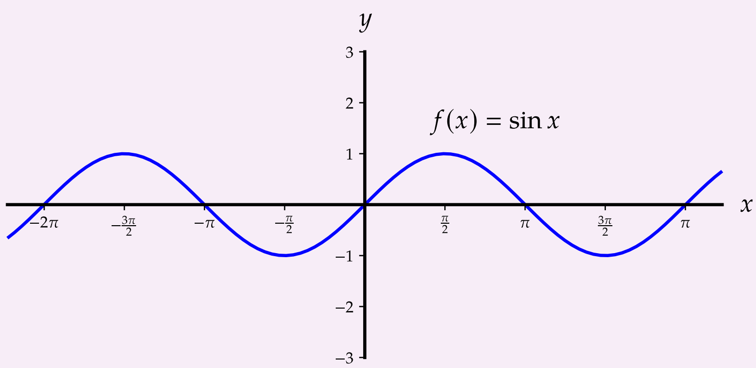
Graphs of Trigonometric Functions

It is not necessary to know the graphs of all six of the trigonometric functions, but three of them are used so often that they are worth knowing.

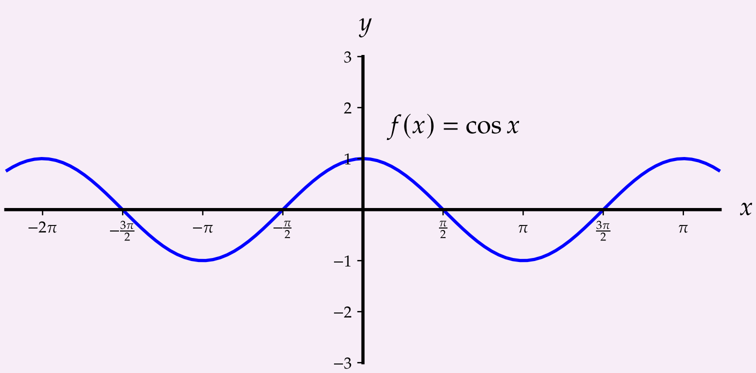
☞ **Need To Know** The graphs of $y = \sin x$, and $y = \cos x$ and $y = \tan x$.

Basic Trigonometric Function Graphs

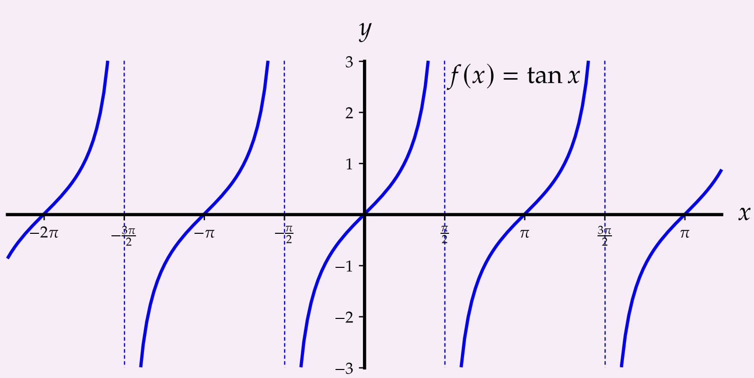
1. $f(x) = \sin x$



2. $f(x) = \cos x$



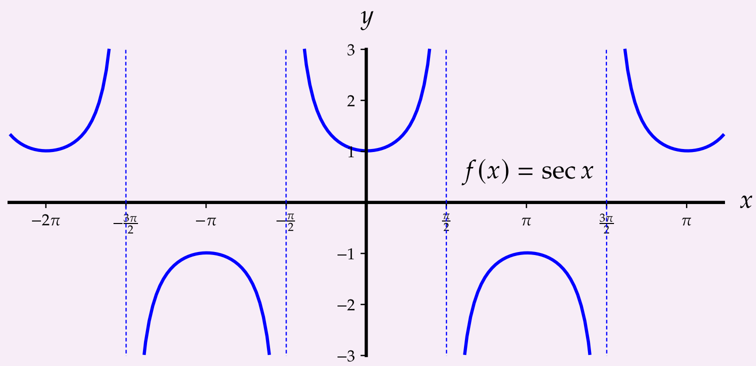
3. $f(x) = \tan x$



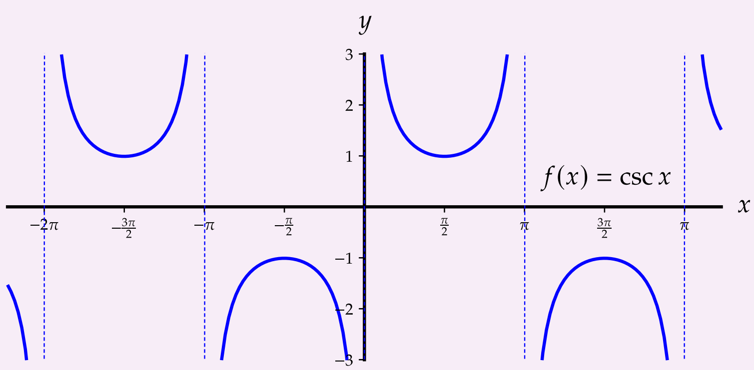
It is worth seeing (though not necessarily memorizing) the graphs of the other three trigonometric functions: $y = \sec x$, and $y = \csc x$ and $y = \cot x$.

Optional: Additional Trigonometric Function Graphs

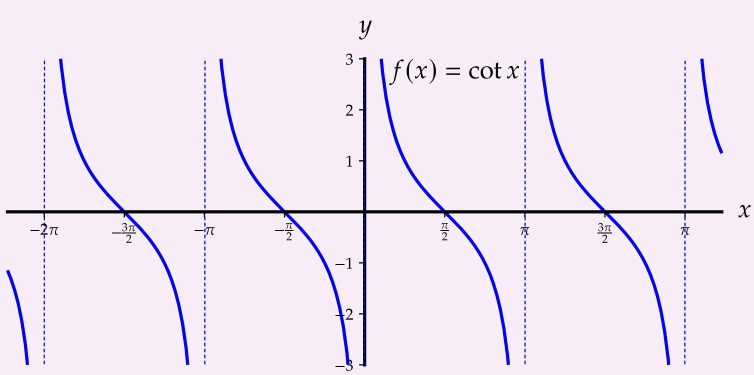
1. $f(x) = \sec x$



2. $f(x) = \csc x$



3. $f(x) = \cot x$



Trigonometric Identities

There are a number of relations, called trigonometric identities, between various of the six trigonometric functions. The most basic trigonometric identities are the following.

Basic Trigonometric Identities

Let x be a real number.

1. $\tan x = \frac{\sin x}{\cos x}$.
2. $\sec x = \frac{1}{\cos x}$.
3. $\csc x = \frac{1}{\sin x}$.
4. $\cot x = \frac{\cos x}{\sin x}$.
5. $\sin(x + 2\pi) = \sin x$.
6. $\cos(x + 2\pi) = \cos x$.
7. $\sin^2 x + \cos^2 x = 1$.

Observe that the identities $\sin(x+2\pi) = \sin x$ and $\cos(x+2\pi) = \cos x$ would also work with 2π replaced by any positive or negative integer multiple of 2π. That is, the value of sine and cosine are unchanged by the addition or subtraction of any integer multiple of 2π to the angle.

The identity $\sin^2 x + \cos^2 x = 1$ is known as the Pythagorean Identity, because it is deduced from the Pythagorean Theorem applied to the triangle in Figure 2 of this chapter (though with the x in this identity corresponding to the angle t in the figure).

Here is an example of using basic trigonometric identities together with the sine and cosine of standard angles to calculate some values of trigonometric functions without a calculator.

Example 3

Evaluate each of the following expressions, without a calculator.

(a) $\sin\left(\frac{25\pi}{4}\right)$

(b) $\sec\left(-\frac{5\pi}{3}\right)$

Solution

(a) We observe that

$$\frac{25\pi}{4} = 6\pi + \frac{\pi}{4} = \frac{\pi}{4} + 3\cdot 2\pi.$$

Because the value of sine is unchanged by the addition or subtraction of any integer multiple of 2π to the angle, we see that

$$\sin\left(\frac{25\pi}{4}\right) = \sin\left(\frac{\pi}{4} + 3\cdot 2\pi\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},$$

where the last equality is from the chart for the sine of standard angles.

(b) We observe that

$$-\frac{5\pi}{3} + 2\pi = \frac{\pi}{3},$$

and therefore

$$-\frac{5\pi}{3} = \frac{\pi}{3} - 2\pi.$$

By using the fact that we can rewrite secant in terms of cosine, together with the fact that the value of cosine is unchanged by the addition or subtraction of any integer multiple of 2π to the angle, we see that

$$\sec\left(-\frac{5\pi}{3}\right) = \frac{1}{\cos\left(-\frac{5\pi}{3}\right)} = \frac{1}{\cos\left(\frac{\pi}{3} - 2\pi\right)} = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = \frac{1}{\frac{1}{2}} = 2,$$

where the equality before the last is from the chart for the cosine of standard angles.

There are a a number of other trigonometric identities that are useful in calculus on occasion. It is not necessary to memorize these formulas, but it is important to know that they exist, and to be able to find them when needed.

Optional: Additional Trigonometric Identities

Let x and y be real numbers.

1. $\sin(-x) = -\sin x$.

2. $\cos(-x) = \cos x$.

3. $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

4. $\sin(x - y) = \sin x \cos y - \cos x \sin y$.

5. $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

6. $\cos(x - y) = \cos x \cos y + \sin x \sin y$.

7. $\sin(2x) = 2 \sin x \cos x$.

8. $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.

9. $\sin^2 x = \dfrac{1 - \cos(2x)}{2}$.

10. $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$.

Here is an example of using trigonometric identities to compute values of sine and cosine, without a calculator.

Example 4

Suppose that angles α and β are between 0 and $\frac{\pi}{2}$, and that $\sin\alpha = \frac{8}{17}$ and $\cos\beta = \frac{7}{25}$. Evaluate each of the following expressions, without a calculator.

(a) $\sin(-\alpha)$

(b) $\cos\alpha$

(c) $\cos(2\beta)$

(d) $\sin(\alpha - \beta)$

Solution Before doing the four parts of this example, we start by finding $\cos\alpha$ and $\sin\beta$, which we do by using the identity $\sin^2 x + \cos^2 x = 1$, where x is replaced by each of α and β. That is, we have

$$\sin^2\alpha + \cos^2\alpha = 1 \quad \text{and} \quad \sin^2\beta + \cos^2\beta = 1,$$

which yields

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{8}{17}\right)^2} = \sqrt{\frac{225}{289}} = \frac{15}{17}$$

and

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{7}{25}\right)^2} = \sqrt{\frac{576}{625}} = \frac{24}{25},$$

where we take the positive square root in each case, because α and β are between 0 and $\frac{\pi}{2}$, and so we know that each of $\cos\alpha$ and $\sin\beta$ is positive.

(a) We use the identity $\sin(-x) = -\sin x$ to compute

$$\sin(-\alpha) = -\sin\alpha = -\frac{8}{17}.$$

(b) We already computed this value above, which is $\cos\alpha = \frac{24}{25}$.

(c) We use the identity $\cos(2x) = \cos^2 x - \sin^2 x$ to compute

$$\cos(2\beta) = \cos^2\beta - \sin^2\beta = \left(\frac{7}{25}\right)^2 - \left(\frac{24}{25}\right)^2 = -\frac{527}{625}.$$

(d) We use the identity $\sin(x-y) = \sin x\cos y - \cos x \sin y$ to compute

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta = \frac{8}{17}\cdot\frac{7}{25} - \frac{15}{17}\cdot\frac{24}{25} = -\frac{304}{425}.$$

EXERCISES – Chapter 7

1–4 ■ Convert each of the following angle given in degrees to radians, without a calculator.

1. $45°$

2. $-30°$

3. $135°$

4. $330°$

5–8 ■ Convert each of the following angle given in radians to degrees, without a calculator.

5. 4π

6. $\frac{\pi}{6}$

7. $-\frac{3\pi}{4}$

8. $\frac{5\pi}{12}$

9–12 ■ Evaluate each expression without a calculator.

9 . $\sin\left(\frac{5\pi}{2}\right)$

10 . $\cos\left(-\frac{\pi}{6}\right)$

11 . $\tan\left(\frac{\pi}{3}\right)$

12 . $\csc\left(\frac{\pi}{2}\right)$

13–20 ■ Suppose that angles α and β are between 0 and $\frac{\pi}{2}$, and that $\sin\alpha = \frac{3}{5}$ and $\cos\beta = \frac{12}{13}$. Evaluate each expression, without a calculator.

13 . $\sin(-\alpha)$

14 . $\cos(-\beta)$

15 . $\cos\alpha$

16 . $\sin\beta$

17 . $\sin(2\alpha)$

18 . $\cos(2\alpha)$

19 . $\sin(\alpha+\beta)$

20 . $\cos(\alpha+\beta)$

21–24 ■ Sketch the graph of each function.

21 . $y = \sin x + 3$

22 . $y = \tan(x - \frac{\pi}{2})$

23 . $y = 3\sec x$

24 . $y = \cos(2x)$

8 Exponential Functions

In Chapter 6 we saw power functions, for example $f(x) = x^2$ and $g(x) = x^\pi$. By contrast, we now discuss functions such as $p(x) = 2^x$ and $q(x) = \pi^x$.

Exponential Functions

The type of function we are now considering is as follows.

Exponential Functions

1. Let a be a positive real number such that $a \neq 1$. The **exponential function** for a is the function $f(x) = a^x$.
2. The **base** of the exponential function $f(x) = a^x$ is the number a.

An example of an exponential function is given by the formula $f(x) = 2^x$. We know what $f(3)$ means, because $f(x) = 2^3 = 2 \cdot 2 \cdot 2 = 8$. We also know what $f(-5)$ and $f(\frac{4}{7})$ means, because we know how to raise the number 2 to negative and fractional powers, as discussed in Chapter 6. What would $f(\pi)$ mean? It equals 2^π, and, although it is not easy to compute that by hand, we already saw what that means when we discussed irrational powers in Chapter 6. Although the discussion in Chapter 6 was for power functions rather than exponential functions, there is no difference when evaluating specific numbers such as 2^π, because the same number arises when we substitute $x = \pi$ into the function $f(x)$ defined by the formula $f(x) = 2^x$ and when we substitute the number $x = 2$ into the function $g(x)$ defined by the formula $g(x) = x^\pi$.

That said, it is important to recognize that exponential functions are very different from power functions. Among the various differences between these two types of functions is the fact that exponential functions grow much faster than polynomial functions as x gets larger. For example, let $f(x)$ be the exponential function defined by the formula $f(x) = 2^x$ and let $g(x)$ be the power function defined by the formula $g(x) = x^2$. We then see that $f(10) = 2^{10} = 1024$, which is much larger than $g(10) = 10^2 = 100$; the difference between the two functions is even more pronounced as we substitute in numbers larger than 10, as the reader can verify.

☞ **Error Warning** Do not confuse exponential functions, which have the form $f(x) = a^x$, and power functions, which have the form $f(x) = x^a$; these two types of functions are very different from each other, with different graphs, different behaviors and different uses.

Exponential functions are extremely useful in many aspects of mathematics and its applications, for example population growth, radioactive decay, differential equations, compound interest, and more.

The Number e and the Function e^x

It is possible to have an exponential function defined by a formula of the form $f(x) = a^x$ for any positive real number a. It turns out, however, that there is one particular number that yields the most useful exponential function, and that is the number e, defined as follows.

The Number e

1. The number e is defined by the formula

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

2. The numerical value of e is $e = 2.718\ldots$.

Similarly to the number π, the decimal expansion of the number e goes on forever without repetition.

The above definition of the number e might seem unmotivated, but the number e turns out to be extremely important in mathematics and its applications. Indeed, the number e is of similar importance as the number π (and, in fact, there is a relation between these two numbers).

You will see why $f(x) = e^x$ works more nicely than exponential functions with other bases when you learn about the derivatives of exponential functions in calculus.

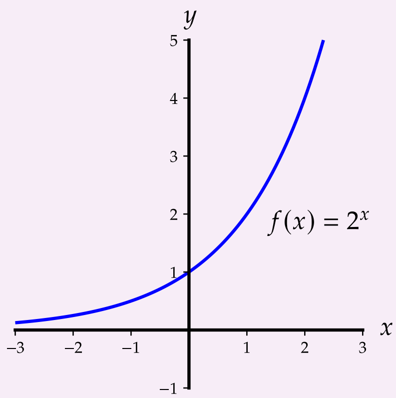
Graphs of Exponential Functions

The graphs of all exponential functions have the same shape, and are just small variations of each other. Specifically, when $a > 1$, all graphs of the form $y = a^x$ are increasing, where the graph increases faster if a is larger; when $0 < a < 1$, all graphs of the form $y = a^x$ are decreasing, where the graph decreases faster if a is smaller; and $y = 1^x$ is just the constant function $y = 1$. It is worth knowing the basic shapes of these graphs when $a > 1$ and when $0 < a < 1$, as well as the graph of the single most important exponential function, which is $y = e^x$.

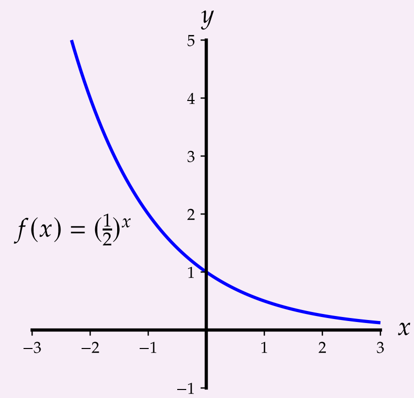
☞ **Need To Know** The graphs of $y = a^x$ when $a > 1$, and $y = a^x$ when $0 < a < 1$, and $y = e^x$.

Exponential Function Graphs

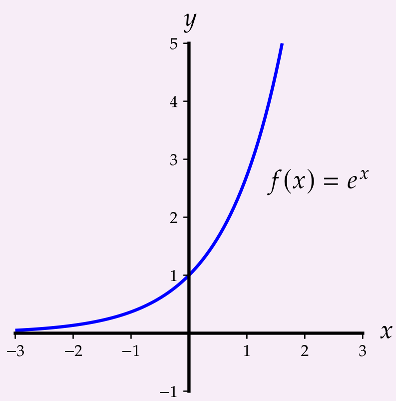
1. $f(x) = a^x$, when $a > 1$



2. $f(x) = a^x$, when $0 < a < 1$



3. $f(x) = e^x$



Properties of Exponentials

Exponential functions have a number of very nice properties, including the following.

Properties of Exponential Functions

Let a and b be positive real numbers, and let x and y be real numbers.

1. $a^{x+y} = a^x a^y$.

2. $a^{x-y} = \frac{a^x}{a^y}$.

3. $(a^x)^y = a^{xy}$.

4. $a^0 = 1$.

5. $(ab)^x = a^x b^x$.

6. $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$.

Here is an example of simplifying an expression using properties of exponential functions.

Example 1

Simplify the expression $\dfrac{(2e^x)^5}{e^{3x}}$, so that it contains only one exponential function.

Solution We use the formulas $(ab)^x = a^x b^x$, and $(a^x)^y = a^{xy}$ and $a^{x-y} = \frac{a^x}{a^y}$ to compute

$$\frac{(2e^x)^5}{e^{3x}} = \frac{2^5(e^x)^5}{e^{3x}} = \frac{32e^{5x}}{e^{3x}} = 32e^{5x-3x} = 32e^{2x}.$$

EXERCISES – Chapter 8

1–2 ■ Simplify the following expressions, so that each contains only one exponential function.

1. $2^{-6x} \cdot (2^{5y})^3$

2. $\left(\dfrac{e^{4x}}{e^{-2x}}\right)^5$

3–6 ■ Sketch the graph of each function.

3. $y = e^{x-2}$

4. $y = e^x - 1$

5. $y = 2e^{x+3}$

6. $y = -e^x - 4$

9 Inverse Functions

The one commonly-used type of function needed for calculus that we have not yet discussed is logarithmic functions. There are a number of ways to define logarithms, but for use in calculus, the most efficient way to define logarithmic functions is via the notion of inverse functions; we will return to logarithmic functions in Chapter 10, but first, in the present chapter, we discuss inverse functions in general.

Inverse Functions

There are many things in life—and in mathematics—where we do something, and we then do another thing that undoes the first thing. For example, if the thing we do is put on our shoes, the thing that undoes that is taking our shoes off. That is, if we do not have shoes on, and if we first put on our shoes, and we then take them off, we are back to where we started; similarly, if we do have shoes on, and if we first take our shoes off, and we then put them on, we are also back to where we started. As such, we can say that the actions of putting shoes on and taking shoes off "cancel each other out," and we can refer to them as "inverse" actions.

In mathematics, the "actions" we are considering are functions. For example, let $f(x) = x^3$. This function takes as input any number (which is denoted "x" in the formula), and produces as output that number raised to the third power. For example, we have $f(2) = 2^3 = 8$. If we wanted to cancel out the result of doing the function $f(x)$, we would need a second function, which would take the output of the $f(x)$ and produce the original input. For example, we would want a function that takes the number 8 as input, and would produce the number 2 as output. The function that evidently works in this case is $g(x) = \sqrt[3]{x}$. For example, we have $g(8) = \sqrt[3]{8} = 2$. More generally, we observe that

$$\sqrt[3]{x^3} = x \quad \text{and} \quad (\sqrt[3]{x})^3 = x,$$

for all real numbers x. We can rewrite this last observation as

$$g(f(x)) = x \quad \text{and} \quad f(g(x)) = x,$$

for all real numbers x. This last relationship is what characterizes a pair of functions as inverse functions, which we state as follows. Observe that in general, we do not refer to "for all real numbers x," but rather refer to "for all appropriate values of x," because not all functions have as their domain all real numbers.

Inverse Functions

Let $f(x)$ and $g(x)$ be functions.

1. The functions $f(x)$ and $g(x)$ are **inverse functions** if

$$g(f(x)) = x \quad \text{and} \quad f(g(x)) = x,$$

for all appropriate values of x.

2. Equivalently, the fact that $f(x)$ and $g(x)$ are inverse functions means

$$y = g(x) \quad \text{if and only if} \quad x = f(y).$$

Using the observation made above about $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$, we see that these two functions are inverse functions.

The most immediate issue to address concerning inverse functions are the questions of whether or not every function has an inverse function, and, if a function does have an inverse function, whether or not the inverse function is unique. We will see shortly an example of a function that does not have an inverse function, so the answer to the first question is no, but it turns out that the answer to the second question is yes.

Existence of Inverse Functions

1. Not every function has an inverse function.

2. If a function $f(x)$ has an inverse function, the inverse function is unique, and is usually denoted $f^{-1}(x)$.

3. If a function $f(x)$ has an inverse function, the inverse function $f^{-1}(x)$ is characterized by

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x,$$

for all appropriate values of x. Equivalently, the inverse function $f^{-1}(x)$ is characterized by

$$y = f^{-1}(x) \quad \text{if and only if} \quad x = f(y).$$

For example, if $f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x}$.

The notation $f^{-1}(x)$ for an inverse function (when it exists) is quite standard, but it is also rather unfortunate, because it is very similar to the notation used for the reciprocal of a number. For example, if we write 3^{-1}, that is equal to $\frac{1}{3}$. However, the notation $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$, even though the use of the symbol "$^{-1}$" looks like we are raising the function to the -1 power; whether "$^{-1}$" means the reciprocal of a number or the inverse function of a function is determined by the context. For example, if $f(x) = x^3$, we observe that $f^{-1}(x) = \sqrt[3]{x}$, which is not the same as $\frac{1}{f(x)} = \frac{1}{x^3}$.

☞ **Error Warning** The notation $f^{-1}(x)$ means the inverse function of $f(x)$; it does *not* mean $\frac{1}{f(x)}$.

Inverse Functions and Graphs

To see which functions have inverse functions and which do not, we turn, as we often do, to graphs of function. Specifically, if a function has an inverse function, the question arises as to whether there is a relation between the graph of a function and the graph of its inverse function.

For example, we see in Figure 1 of this chapter the graph of the function $f(x) = x^3$, and we see in Figure 2 of this chapter the graph of the the inverse function $f^{-1}(x) = \sqrt[3]{x}$. A comparison of the two graphs show

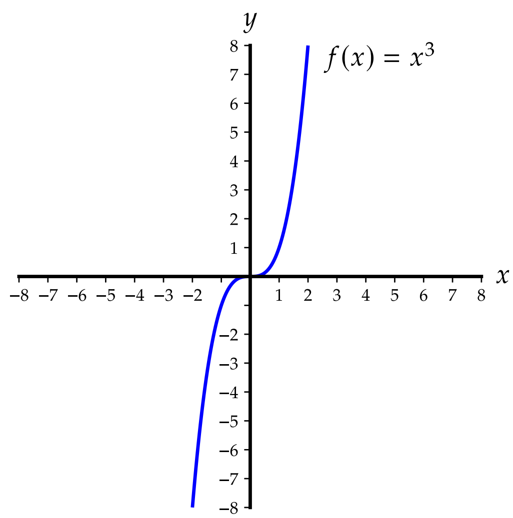


Figure 1: Graph of $f(x) = x^3$

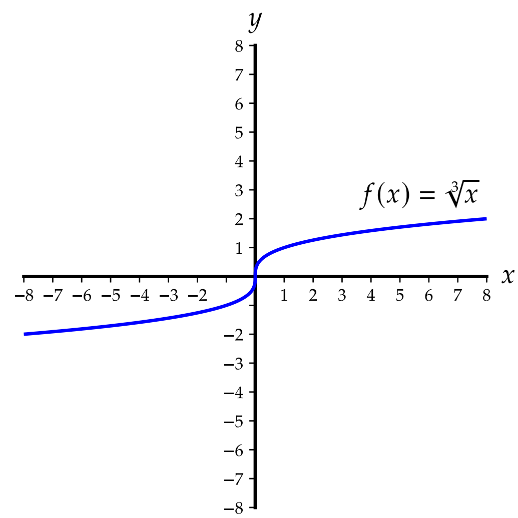


Figure 2: Graph of $f^{-1}(x) = \sqrt[3]{x}$

that the graph of $f^{-1}(x) = \sqrt[3]{x}$ is the result flipping the graph of $f(x) = x^3$ in a line through the origin with slope 1, which is the line $y = x$; see Figure 3 of this chapter, where the graph of $f(x) = x^3$ is shown dotted, to keep the two graphs distinct.

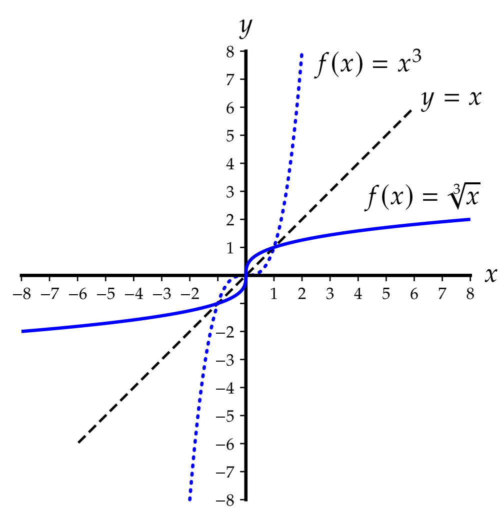


Figure 3: Graphs of $f(x) = x^3$ and $f^{-1}(x) = \sqrt[3]{x}$

We now state the above idea in general. Let $f(x)$ be a function. Suppose that $f(x)$ has an inverse function, denoted $f^{-1}(x)$. The key to understand the relation between the graph $f(x)$ and the graph of $f^{-1}(x)$ is the fact that

$$y = f^{-1}(x) \quad \text{if and only if} \quad x = f(y).$$

That tells us that the graph of $y = f^{-1}(x)$ is the same as the graph of $x = f(y)$, where the latter is the result of taking $y = f(x)$ and interchanging the roles of x and y.

Geometrically, interchanging the roles of x and y means interchanging the x-axis and the y-axis, and that in turn means reflecting (also called "flipping") the plane in the line $y = x$, which is the line though the origin with slope 1. Hence, the graph of $y = f^{-1}(x)$ is obtained from the graph of $y = f(x)$ by taking the graph of $y = f(x)$ and reflecting it in the line $y = x$.

Graphs of Inverse Functions

Let $f(x)$ and $g(x)$ be functions. If $f(x)$ and $g(x)$ are inverse functions, then the graph of $y = g(x)$ is obtained by reflecting the graph of $y = f(x)$ in the line $y = x$.

We can now use the above observation about graphs of inverse functions to test whether or not a given function has an inverse function.

For example, we see in Figure 4 of this chapter the graph of the function $f(x) = x^2$, as well as the line $y = x$, and we see in Figure 5 of this chapter the result of reflecting the graph of $f(x) = x^2$ in the line $y = x$. We now recall the Vertical Line Test, discussed in Chapter 3, and we observe that the curve seen in Figure 5 does not pass this test, because there are vertical lines that intersect the curve in more than one point, for example the dashed line shown in the figure. As such we see that the curve in Figure 5 is not the graph of a function. However, we know that if the function $f(x) = x^2$ had an inverse function, the graph of the inverse function would be the curve shown in Figure 5, and because this curve is not the graph of a function, the only possible conclusion is that $f(x) = x^2$ does not have an inverse function.

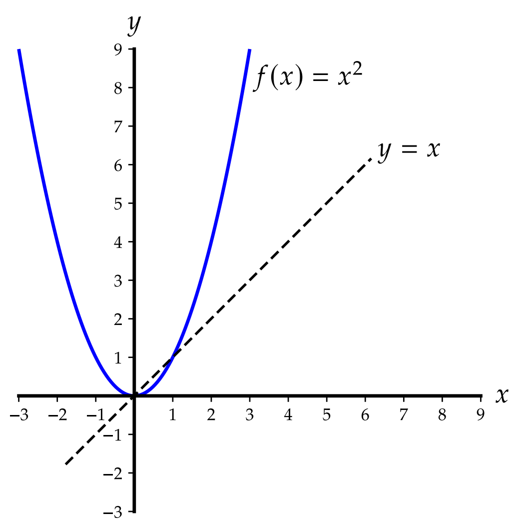


Figure 4: Graph of $f(x) = x^2$

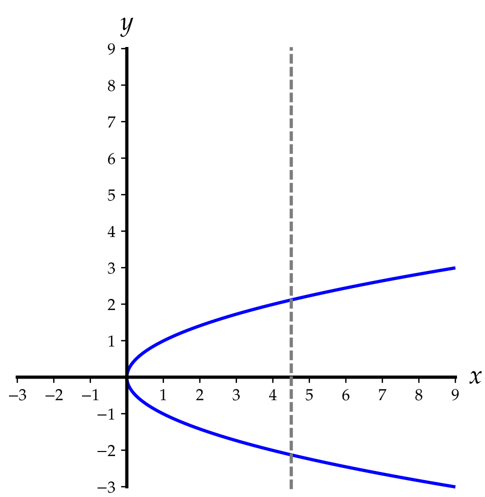


Figure 5: Reflection of graph of $f(x) = x^2$

It might seem that in order to test if a function has an inverse function, we would always have to do what we did above, which is to draw the graph of the function, to draw the reflection of the graph of the function in the line $y = x$, and check whether or not the reflection of the graph passes the Vertical Line Test. There is, fortunately, a short cut to doing the above procedure, which is seen in Figure 6 of this chapter, which is the result of taking Figure 5 of this chapter and reflecting it back in the line $y = x$, so that the graph becomes the parabola $y = x^2$ again in Figure 6, and the line that was vertical in Figure 5 becomes horizontal in Figure 6.

We then observe that the reflected curve in Figure 5 passes the Vertical Line Test precisely if the graph in Figure 6 is such that any horizontal line intersects the graph in at most one point (which is not true for the graph of $y = x^2$). Any curve that satisfies this condition with regard to horizontal lines is said to pass the Horizontal Line Test, and this test is what we use to determine if a function has an inverse function; the advantage of the Horizontal Line Test is that we do not have to draw the reflection of the graph in the line $y = x$.

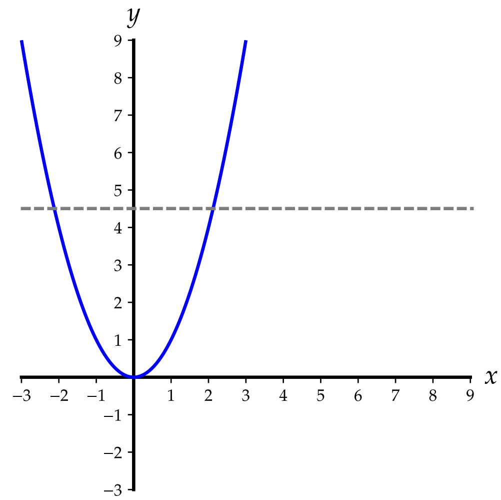


Figure 6: Graphs of $f(x) = x^2$ with horizontal line

Horizontal Line Test

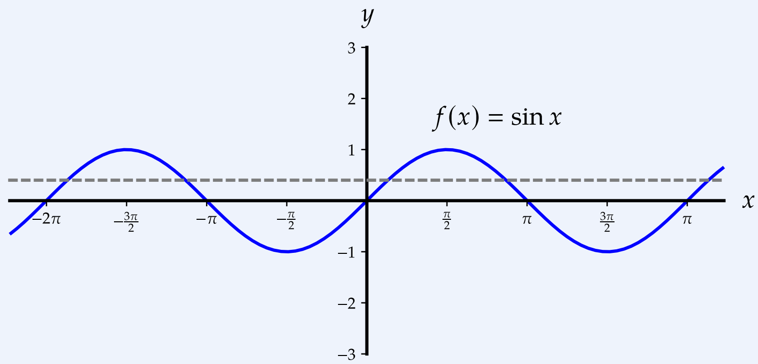
1. A curve in the plane passes the **Horizontal Line Test** if every horizontal line in the plane intersects the curve in at most one point.
2. A function has an inverse function if and only if the graph of the function satisfies the Horizontal Line Test.

Here are examples of using the Horizontal Line Test.

Example 1

Determine whether the function $f(x) = \sin x$ has an inverse function.

Solution As we see in the following figures, there are horizontal lines that intersect the graph of $f(x) = \sin x$ in more than one point; in fact, the horizontal line in the figure intersects the graph in infinitely many points. Hence, the function does not pass the Horizontal Line Test, and so we know that $f(x) = \sin x$ does not have an inverse function.



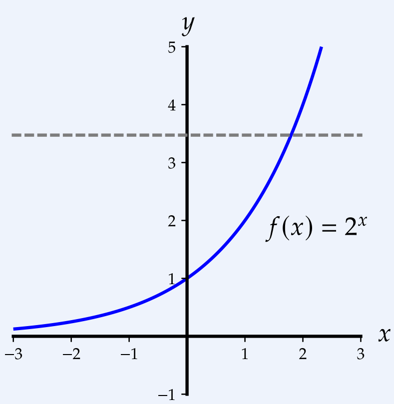
Graph of $f(x) = \sin x$

As the reader might see in a calculus class, where inverse trigonometric functions are discussed, the the problem of $f(x) = \sin x$ not having an inverse function is resolved by restricting the function to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, on which the Horizontal Line Test is passed.

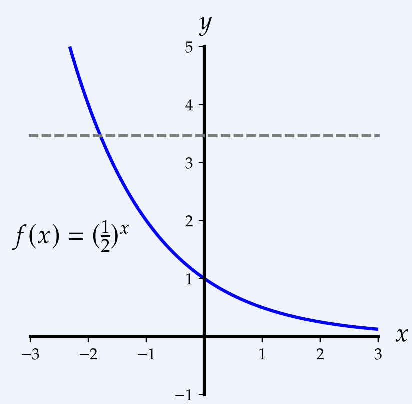
Example 2

Let a be a positive real number such that $a \neq 1$. Determine whether the function $f(x) = a^x$ has an inverse function.

Solution If $a > 1$, then the graph of $f(x) = a^x$ looks similar to the graph of $f(x) = 2^x$ seen below on the left, and if $0 < a < 1$, then the graph of $f(x) = a^x$ looks similar to the graph of $f(x) = \left(\frac{1}{2}\right)^x$ seen below on the right. In both cases, it is evident that every horizontal line in the plane either intersects the graph once (if the line is above the x-axis) or not at all (if the line is on or below the x-axis), and the function passes the Horizontal Line Test. Therefore $f(x) = a^x$ has an inverse function when $a \neq 1$. (When $a = 1$, the function $f(x) = a^x$ is the function that is constantly 1, and it does not have an inverse function; of course, it is silly to think of the function that is constantly 1 as $f(x) = 1^x$, rather than simply $f(x) = 1$.)



Graph of $f(x) = 2^x$



Graph of $f(x) = \left(\frac{1}{2}\right)^x$

Finally, we note that whereas the Horizontal Line Test is very useful in determining whether or not a given function has an inverse function, this test is useful in practice only for those functions for which we can find their graphs. Fortunately, there are other ways of showing that a function has an inverse function, which do not involve finding the graph of the function; one such approach uses calculus, where we see a method of verifying that a function is increasing everywhere or decreasing everywhere, which would imply that the function has an inverse function.

Finding Inverse Functions

The Horizontal Line Test can tell us whether or not a function has an inverse function (as long as we can graph the function), but, if the function does have an inverse function, the Horizontal Line Test does not tell us how to find a formula for that inverse function.

In theory there is a simple method for finding a formula for the inverse function in those cases where an inverse function exists, though in practice it is somewhat rare that the method can be easily carried out.

Let $f(x)$ be a function. Suppose that we know that $f(x)$ has an inverse function, denoted $f^{-1}(x)$. Recall that one way of characterizing what it means that $f(x)$ and $f^{-1}(x)$ are inverse function is that

$$y = f^{-1}(x) \quad \text{if and only if} \quad x = f(y).$$

What that says for our present purpose is that if we want to find a formula for $f^{-1}(x)$, we start with the function $y = f(x)$, we then switch the roles of x and y, yielding $x = f(y)$, and we then solve this equation for y, which will yield $y = f^{-1}(x)$. The fact that this method works is really the same as the fact that the graph of $y = f^{-1}(x)$ is the result of taking the graph of $y = f(x)$ and reflecting it in the line $y = x$, because reflecting in the line $y = x$ interchanges the x-axis and y-axis, which means switching the roles of x and y, and that is precisely what we do when we go from $y = f(x)$ to $x = f(x)$.

Finding Inverse Functions

Suppose that the function $f(x)$ has an inverse function, denoted $f^{-1}(x)$. A formula for the inverse function can be found by writing $y = f(x)$, then switching switch the roles of x and y, yielding $x = f(y)$, and then solving this equation for y in terms of x, which will yield $y = f^{-1}(x)$.

Here are two examples of finding, or attempting to find, inverse functions.

Example 3

Find the inverse function of the function $f(x) = 3x + 7$.

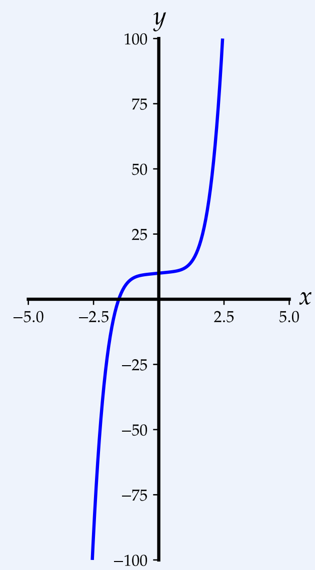
Solution The graph of $f(x)$ is a line with slope $m = 3$, and it is evident that this graph satisfies the Horizontal Line Test. It follows that this function has an inverse function.

To find the inverse function, we start by writing $y = f(x)$, which is $y = 3x + 7$. Next, we switch the roles of x and y, yielding $x = 3y + 7$. Finally, we solve this last equation for y in terms of x, which is done by first obtaining $x - 7 = 3y$, and then $y = \frac{1}{3}x - \frac{7}{3}$. We deduce that $f^{-1}(x) = \frac{1}{3}x - \frac{7}{3}$.

Example 4

Find the inverse function of the function $g(x) = x^5 + x + 10$.

Solution The graph of the function can be seen below. Using the Horizontal Line Test, we see that this function has an inverse function; a more precise argument that the function has an inverse function can be obtained when the reader learns some calculus, by showing that the function is increasing everywhere.



Graph of $f(x) = x^5 + x + 10$

To try to find the inverse function, we would start by writing $y = g(x)$, which is $y = x^5 + x + 10$. Next, we would switch the roles of x and y, yielding $x = y^5 + y + 10$. We would then want to solve this last equation for y in terms of x, but unfortunately that cannot be done. Hence, whereas the function $g(x) = x^5 + x + 10$ has an inverse function in theory, we are not able to find a formula for the inverse function.

EXERCISES – Chapter 9

1–6 ■ Determine whether or not each of the following functions has an inverse function by using its graph.

1. $f(x) = 4x + 12$

2. $g(x) = 2|x| + 4$

3. $w(x) = x^2 - 4$

4. $c(x) = |x - 5|$

5. $h(x) = \dfrac{1}{x}$

6. $q(x) = \begin{cases} 6x - 3x^2, & \text{if } x \geq 1 \\ -x + 4, & \text{if } x < 1. \end{cases}$

7–8 ■ Each of the following functions has an inverse function; *there is no need to verify that*; find a formula for the inverse function for each of the following functions.

7 . $f(x) = 4x + 12$

8 . $g(x) = 2x^3 + 5$

9 . $r(x) = \sqrt[5]{x+1}$

10 . $q(x) = e^{2x}$

10 Logarithmic Functions

Some people find logarithmic functions to be a bit mysterious, due people to the somewhat indirect way in which they are usually defined, but in fact logarithmic functions are just normal functions, which happen to very useful.

Another misconception about logarithms is related to their history. They originally arose in order to simplify lengthy numerical calculations when such calculations where done by hand (which was most of human history), and, the thinking goes, now that we have calculators and computers, perhaps we should not be interested in logarithms any more. In fact, a mathematical—or other—idea can arise for one reason, but then turn out to be useful for other reasons, even when the original reason is no longer relevant, and logarithms are an example of that. Not only are logarithms useful in a variety of places in mathematics and its applications, but, given the relation between exponential functions and logarithmic functions, any place where exponential functions are used, logarithms are often needed too.

There are a number of ways to think about logarithms. For use in calculus, a very common way to define logarithmic functions is via the notion of inverse functions, as discussed in Chapter 9.

Definition of Logarithmic Functions

Let a be a positive real number such that $a \neq 1$. We saw in Example 2 of Chapter 9 that a^x has an inverse function. The name we give to this inverse function is the logarithmic function with base a.

Logarithmic Functions

Let a be a positive real number such that $a \neq 1$.

1. The **logarithmic function with base** a is the function $f(x) = \log_a x$ that is defined to be the inverse function of $g(x) = a^x$.
2. The **base** of the logarithmic function $f(x) = \log_a x$ is the number a.
3. The domain of $\log_a x$ is $(0, \infty)$.

It is important to note that for each positive real number a such that $a \neq 1$, the is a logarithmic function with that base. For example, there are $\log_2 x$, $\log_3 x$, $\log_\pi x$, and so on. Though we can in theory have logarithmic functions when $0 < a < 1$, for example $\log_{0.5} x$, in practice the logarithmic functions that are used in calculus and its applications typically have $a > 1$.

If we recall the basic properties of inverse functions, as discussed in Chapter 9, we immediately deduce the following defining properties of logarithmic functions, because the latter are defined as inverse functions.

Defining Properties of Logarithmic Functions

Let a be a positive real number such that $a \neq 1$.

1. The function $\log_a x$ is characterized by

$$\log_a(a^x) = x \quad \text{and} \quad a^{\log_a x} = x,$$

   for all appropriate values of x.

2. Equivalently, the function $\log_a x$ is characterized by

$$\log_a x = y \quad \text{if and only if} \quad a^y = x.$$

Here is an example of using the defining properties of logarithmic functions.

Example 1

Simplify the expression $\log_5 \frac{1}{\sqrt{5}}$, without a calculator.

Solution By the definition of the function $\log_5 x$, we know that

$$\log_5 \frac{1}{\sqrt{5}} = y$$

for some number y if and only if

$$5^y = \frac{1}{\sqrt{5}}.$$

Using basic facts about power functions, we know that

$$\frac{1}{\sqrt{5}} = 5^{-\frac{1}{2}},$$

which means that we can take y to be $-\frac{1}{2}$. It follows that

$$\log_5 \frac{1}{\sqrt{5}} = -\frac{1}{2}.$$

Natural and Common Logarithmic Functions

Just as $f(x) = e^x$ is the most important exponential functions, so too the logarithmic function with base e is the most important logarithmic function; this logarithmic function is so widely used it has its own name and notation.

Natural Logarithm Function

1. The **natural logarithm function** is the function $f(x) = \ln x$, where $\ln x$ is an abbreviation for $\log_e x$; that is, the function $f(x) = \ln x$ is defined to be the inverse function of $g(x) = e^x$.
2. The **base** of the natural logarithm function $f(x) = \ln x$ is the number e.
3. The domain of $\ln x$ is $(0, \infty)$.

As with all other logarithmic functions, so too the natural logarithm function has defining properties that follow from its definition as an inverse function.

Defining Properties of the Natural Logarithm Function

1. The function $\ln x$ is characterized by

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x,$$

for all appropriate values of x.

2. Equivalently, the function $\ln x$ is characterized by

$$\ln x = y \quad \text{if and only if} \quad e^y = x.$$

Another logarithmic function with its own name is the "common logarithm," which is the logarithmic function that has base 10. The common logarithmic function is often denoted $\log x$. Whereas the common logarithm was very important when logarithms were used to help do numerical calculations, the common logarithm is not as important today.

Graphs of Logarithmic Functions

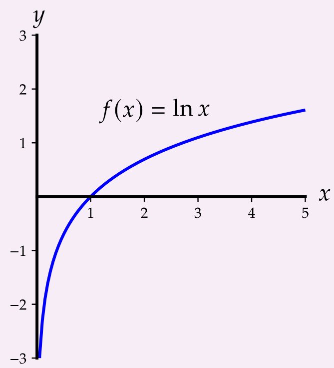
Because logarithmic functions are the inverse functions of exponential functions, the graphs of logarithmic functions are obtained by reflecting the graphs of the corresponding exponential functions in the line $y = x$.

It follows that the graphs of all logarithmic functions have the same shape, and are just small variations of each other. Specifically, when $a > 1$, all graphs of the form $y = \log_a x$ are increasing, where the graph increases more slowly if a is larger; when $0 < a < 1$, all graphs of the form $y = \log_a x$ are decreasing, where the graph decreases more slowly if a is smaller. Because the natural logarithm function is by far the most widely-used logarithmic function, its graph is the important one to know.

☞ **Need To Know** The graph of $y = \ln x$.

Logarithmic Function Graphs

1. $f(x) = \ln x$



Properties of Logarithms

Logarithmic functions have a number of very nice properties, including the following. We state these properties first in general, and then in the specific case of the natural logarithm function (which is redundant, but is so important that it is worth stating explicitly).

Properties of Logarithmic Functions

Let a be a positive real number such that $a \neq 1$, and let x and y positive real numbers.

1. $\log_a(xy) = \log_a x + \log_a y$.
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$.
3. $\log_a x^r = r \log_a x$.
4. $\log_a 1 = 0$.

Properties of the Natural Logarithm Function

Let x and y positive real numbers.

1. $\ln(xy) = \ln x + \ln y$.
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$.
3. $\ln x^r = r \ln x$.
4. $\ln 1 = 0$.

Here are three examples of using properties of logarithms.

Example 2

Rewrite the expression $6\log_3 a + \log_3(a+\sqrt{a})$ as a single logarithm.

Solution We use the formula $\log_a(xy) = \log_a x + \log_a y$ and $\log_a x^r = r\log_a x$ with $a = 3$ to compute

$$6\log_3 a + \log_3(a+\sqrt{a}) = \log_3 a^6 + \log_3(a+\sqrt{a}) = \log_3 a^6(a+\sqrt{a}) = \log_3(a^7 + a^{\frac{13}{2}}).$$

Example 3

Rewrite the expression $\ln\left(\frac{v^5}{\sqrt[3]{w}}\right)$ in terms of $\ln v$ and $\ln w$.

Solution We use the formulas $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ and $\ln x^r = r\ln x$ to compute

$$\ln\left(\frac{v^5}{\sqrt[3]{w}}\right) = \ln v^5 - \ln\sqrt[3]{w} = \ln v^5 - \ln w^{\frac{1}{3}} = 5\ln v - \frac{1}{3}\ln w.$$

Example 4

Simplify the expression $e^{5\ln p + \ln m}$.

Solution We use the formulas $\ln(xy) = \ln x + \ln y$, and $\ln x^r = r\ln x$ and $e^{\ln x} = x$ to compute

$$e^{5\ln p+\ln m} = e^{\ln p^5 + \ln m} = e^{\ln(p^5 m)} = p^5 m.$$

One of the uses of logarithmic functions is to solve equations involving exponential functions; similarly, exponential functions are used to solve equations involving logarithmic functions. The following example shows some instances of such equations.

Example 5

Solve each of the following equations.

(a) $e^{x^3}e^5 = 2$

(b) $\ln(x^5) + \ln(2x) = 9$

Solution

(a) We start with the equation

$$e^{x^3}e^5 = 2,$$

and we then use the formula $a^{x+y} = a^x a^y$ with $a = e$, which allows us to rewrite the equation as

$$e^{x^3+5} = 2.$$

Next, we take ln of both sides, obtaining

$$\ln(e^{x^3+5}) = \ln 2.$$

We then use the formula $\ln(e^x) = x$ to simplify the equation as

$$x^3 + 5 = \ln 2.$$

It follows that

$$x^3 = \ln 2 - 5,$$

and hence

$$x = \sqrt[3]{\ln 2 - 5}.$$

(b) We start with the equation

$$\ln(x^5) + \ln(2x) = 9,$$

and we then use the formula $\ln(xy) = \ln x + \ln y$, which allows us to rewrite the equation as

$$\ln(x^5 \cdot 2x) = 9,$$

which is simplified to

$$\ln(2x^6) = 9.$$

Next, we raise e to the power of both sides, obtaining

$$e^{\ln(2x^6)} = e^9.$$

We then use the formula $e^{\ln x} = x$ to simplify the equation as

$$2x^6 = e^9.$$

It follows that

$$x^6 = \frac{e^9}{2},$$

and hence

$$x = \pm\sqrt[6]{\frac{e^9}{2}}.$$

Relations Between Logarithms

For each positive real number a such that $n \neq 1$, there is a logarithmic function $f(x) = \log_a x$. It turns out, however, that any two such logarithmic functions are related, and, in particular, any logarithmic function is related to the natural logarithm function, as follows.

Relations Between Logarithms

Let a and b be a positive real number such that $a \neq 1$ and $b \neq 1$, and let x be a positive real number.

1. $\log_a x = \dfrac{\log_b x}{\log_b a}$.

2. $\log_a x = \dfrac{\ln x}{\ln a}$.

Because $\log_b a$ and $\ln a$ in the above formulas are constants, we see that any two logarithmic functions are constant multiples of each other. For example, have $\log_{10} x = \frac{1}{\ln 10} \ln x$.

EXERCISES – Chapter 10

1–4 ■ Simplify each expression, without a calculator.

1. $\log_3 9$

2. $\log_9 3$

3. $\log_2 \dfrac{1}{32}$

4. $\log_2 \sqrt[4]{2}$

5–8 ■ Rewrite each expression as a single logarithm.

5. $\log_5 p^4 - \log_5 p$

6. $3 \ln x + 4 \ln x$

7. $5 \log_2 x - 3 \log_2 z$

8. $\dfrac{1}{2} \ln a - \dfrac{1}{3} \ln b$

9–12 ■ Rewrite each expression in terms of $\ln x$, $\ln y$ and $\ln z$.

9. $\ln \sqrt[5]{x}$

10. $\ln(x^2 y^5 z)$

11. $\ln \left(\dfrac{x^3}{z^2} \right)$

12. $\ln \sqrt{x^4 y^3}$

13–16 ■ Simplify each expression.

13. $e^{\ln(2a+b)}$

14. $\ln(e^{xy})$

15. $e^{2 \ln x + 3 \ln w}$

16. $\ln(e^{m^2} e^{3n})$

17–20 ■ Solve each equation.

17. $e^{x+5} = 3$

18. $\ln(x^2 - 3) = 4$

19. $e^{x^2 - 2x} e^{2x+1} = 8$

20. $\ln(x + 1) - \ln x = 1$

21–24 ■ Sketch the graph of each function.

21. $y = \ln(x + 1)$

22. $y = \ln x + 1$

23. $y = 2 \ln(x - 3)$

24. $y = -\ln x + 2$