Read and comment upon the following attempted proofs as if you are the professor in a class that teaches proofs. Your comments should indicate what is wrong, and give suggestions for improvement. Some of these proofs are wildly incorrect, and others are mostly correct, though with small errors of content or writing style. Photocopy the proofs or download them at `http://math.bard.edu/bloch/you_are_the_prof.pdf`, take a red pen and go at them mercilessly.

Exercise 8.8.1. [Same as Exercise 2.5.5 (1)] Prove or give a counterexample to the following statement: For each real number x, there exists a real number y such that $e^x - y > 0$.

> ***Proof (A).*** The statement is true. For any x we can choose $y = 0$. Since $0 < e^x$ for all x, we have that for all x we can choose a y such that $e^x - y > 0$.

> ***Proof (B).*** The statement is true. Let $y = 0$. Since $x \in \mathbb{R}$, therefore $e^x - y > 0$ for each x.

> ***Proof (C).*** The statement is true. Let $x \in \mathbb{R}$. For all x, $e^x > 0$. Let $y = 0$. For all x, $e^x - y > 0$.

Exercise 8.8.2. [Same as Exercise 5.3.4 (1)] Let A and B be sets, and let $f: A \to B$ be a function. Let $\sim$ be the relation on A defined by $x \sim y$ if and only if $f(x) = f(y)$, for all $x, y \in A$. Prove that $\sim$ is an equivalence relation.

> ***Proof (A).*** We will prove that $\sim$ is reflexive. Suppose that $x \sim x$. Then $f(x) = f(x)$, for $x \in A$. Suppose that $f(x) = f(x)$, then $x \sim x$. Hence $\sim$ is reflexive.

> ***Proof (B).*** We will prove that $\sim$ is symmetric. Suppose that $f(x) = f(y)$. Then $f(y) = f(x)$. So $x \sim y$ and $y \sim x$. Now suppose that $f(x) \neq f(y)$. Then $f(y) \neq f(x)$. So $x \not\sim y$ and $y \not\sim x$. So $x \sim y$ if and only if $y \sim x$; that is, $\sim$ is symmetric.

> ***Proof (C).*** We will prove that $\sim$ is transitive. Since $f(x) = f(x)$, $x \sim x$ so $\sim$ is reflexive. If $z \in A$ and $f(x) = f(y) = f(z)$ then $x \sim y$ and $y \sim z$ implies $x \sim z$, so $\sim$ is transitive.

Exercise 8.8.3. [Same as Exercise 3.3.11] Let X be a set, and let $A, B, C \subseteq X$ be subsets. Suppose that $A \cap B = A \cap C$, and that $(X - A) \cap B = (X - A) \cap C$. Prove that $B = C$.

Proof (A). First, we show that $B \subseteq C$. Let $p \in A \cap B$. This means that $p \in A$ and $p \in B$. Since $A \cap B = A \cap C$, $A \cap B \subseteq A \cap C$ and $A \cap C \subseteq A \cap B$. That means that $p \in A \cap B$ implies $p \in A \cap C$. Since $p \in A \cap C$, it follows that $p \in C$. Because $p \in A \cap B$ means that $p \in B$, $p \in B$ implies $p \in C$. Therefore $B \subseteq C$.

Second, we show that $C \subseteq B$. Let $p \in (X - A) \cap C$. This means that $p \in C$. Since $(X - A) \cap C = (X - A) \cap B$, we know that $(X - A) \cap C \subseteq (X - A) \cap B$. This means that $p \in (X - A) \cap C$ implies $p \in (X - A) \cap B$. Since $p \in (X - A) \cap B$, it can be said that $p \in B$. Since $p \in (X - A) \cap C$ means $p \in C$, $p \in C$ implies $p \in B$. Therefore $C \subseteq B$.

We have shown both that $B \subseteq C$ and $C \subseteq B$. Therefore, $B = C$.

Proof (B). First, I will show that $B \subseteq C$. Let $x \in B$.

Considering $A \cap B = A \cap C$, $x \in A \cap B$ implies that $x \in A \cap C$. According to Theorem 3.3.3 (1), $A \cap B \subseteq B$ and $A \cap C \subseteq C$. Since $A \cap B = A \cap C$, it can be said that $A \cap B \subseteq C$. Since $x \in A \cap B$ and therefore $x \in B$, and $A \cap B \subseteq C$, x is therefore also an element of C in the case that $A \cap B = A \cap C$.

Likewise, $(X - A) \cap B = (X - A) \cap C$ implies that a given element x exists in $(X - A) \cap B$ and $(X - A) \cap C$. Therefore $x \notin A$ and $x \in B$. Also, according to Theorem 3.3.3, $(X - A) \cap C \subseteq C$. Since $(X - A) \cap C = (X - A) \cap B$, then $(X - A) \cap B \subseteq C$. This implies that there is an element $x \in (X - A) \cap B$ and $x \in C$. The phrase $x \in (X - A) \cap B$ can be further broken down to $x \notin A$ and $x \in B$. Therefore $B \subseteq C$, regardless of whether $x \in A$ or $x \notin A$.

To prove that $C \subseteq B$, it suffices to show that element $y \in C$ implies $y \in B$. Take $(X - A) \cap B = (X - A) \cap C$. Then $y \in (X - A) \cap B$, and thus $y \in B$ and $y \notin A$. According to Theorem 3.3.3, $(X - A) \cap B \subseteq B$. Since $(X - A) \cap B = (X - A) \cap C$, then $(X - A) \cap C \subseteq B$. This implies that $y \notin A$ and $y \in C$. Also $y \in B$. Therefore, $C \subseteq B$. If $B \subseteq C$ and $C \subseteq B$, then $B = C$.

Proof (C). Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Also let $x \in A \cap C$, then $x \in A$ and $x \in C$. This shows that $x \in A$, B and C. Since $(X - A) \cap B = (X - A) \cap C$, then $x \in X$ and $x \notin A$ and $x \in B$ and $x \in X$ and $x \notin A$ and $x \in C$ are equivalent. Therefore $x \in X$, B, and C and $x \notin A$. By Theorem 3.3.3, $A \cap B \subseteq A$ and $A \cap B \subseteq B$, and $A \cap C \subseteq A$ and $A \cap C \subseteq C$. However, since $x \notin A$, then $B = C$.

Exercise 8.8.4. [Same as Exercise 4.2.11] Let A and B be sets, let $P, Q \subseteq A$ be subsets and let $f : A \to B$ be a function.

(1) Prove that $f(P) - f(Q) \subseteq f(P - Q)$.
(2) Is it necessarily the case that $f(P - Q) \subseteq f(P) - f(Q)$? Give a proof or a counterexample.

Proof (A).

(1). Let $b \in B$. By Definition 4.2.1, there is some $x \in f(P) - f(Q)$ such that $b \in B$ is also $b \in f(P) - f(Q)$. By definition of $f(P)$, there is some $p \in P$ and $q \in Q$ such that $b \in f(p)$ and $b \notin f(q)$. This means that there is $a \in A$ such that $a \in p$ and $a \notin q$. Thus $a \in p - q$, and therefore $b \in f(p - q)$. Thus, $x \in f(P - Q)$. Thus $f(P) - f(Q) \subseteq f(P - Q)$.

(2). It is also the case that $f(P - Q) \subseteq f(P) - f(Q)$. Let $b \in B$. By Definition 4.2.1, there exists x such that $x \in f(P - Q)$ in which there exist elements $p \in P$ and $q \in Q$ such that $b \in f(p - q)$. Therefore $b \in f(p)$ and $b \notin f(q)$. Therefore $x \in f(P)$ and $x \notin f(Q)$, and thus $x \in f(P) - f(Q)$.

Therefore $x \in f(P - Q)$ implies that $x \in f(P) - f(Q)$. Thus $f(P - Q) \subseteq f(P) - f(Q)$.

Proof (B).

(1). Let $f(x) \in f(P) - f(Q)$. This implies that $f(x) \in f(P)$ and $f(x) \notin f(Q)$, and thus that $x \in P$ and $x \notin Q$. It follows that $x \in P - Q$ and that $f(x) \in f(P - Q)$.

(2). Let $f(y) \in f(P - Q)$. This implies that $y \in P - Q$ and thus that $y \in P$ and $y \notin Q$. Hence, $f(y) \in f(P)$ and $f(y) \notin f(Q)$, and $f(y) \in f(P) - f(Q)$.

Proof (C).

(1). Let $f : A \to B$ be a function, and let $x \in f(P) - f(Q)$. So, by the definition of the image, it follows that $f(x) \in P$ and $f(x) \notin Q$. Therefore by definition of the image, $f(x) \in P - Q$, and $x \in f(P - Q)$. So $f(P) - f(Q) \subseteq f(P - Q)$.

(2). Now let $x \in f(P - Q)$. So, by the definition of the image, $f(x) \in P - Q$. This means that $f(x) \in P$ and $f(x) \notin Q$. This implies that $x \in f(P)$ and $x \notin f(Q)$. Therefore $x \in f(P) - f(Q)$. It follows that $f(P - Q) \subseteq f(P) - f(Q)$.

Proof (D).

(1). Let $k \in f(P) - f(Q)$. This means that $k \in f(P)$ and $k \notin f(Q)$. By Definition 4.2.1 this means that $k = f(p)$ for all $p \in P$ and that $k \neq f(q)$ for any $q \in Q$. Since $k = f(p)$ for all $p \in P$ and $k \neq f(q)$ for all $q \in Q$, it follows that $f(p) \neq f(q)$. Because of Definition 4.2.1, which states that in a function $f\colon A \to B$ each $a \in A$ maps to only one $b \in B$ and each $b \in B$ has only one $a \in A$ that maps to it, the fact that $f(p) \neq f(q)$ implies that $p \neq q$. Because $p \neq q$ for all $q \in Q$, it follows that $p \notin Q$. Since $p \in P$ and $p \notin Q$, we can derive that $p \in P - Q$. This means that $f(p) \in f(P-Q)$, which means that $k \in f(P-Q)$. Since $k \in f(P) - f(Q)$ implies $k \in f(P-Q)$, it follows that $f(P) - f(Q) \subseteq f(P-Q)$.

(2). Let $b \in f(P-Q)$. By Definition 4.2.1, there is some $r \in P - Q$ such that $b = f(r)$. Also, $r \in P$ and $r \notin Q$ by Definition 3.3.6. Because $b \in B$, $b = f(r)$, and $r \in P$, we know from Definition 4.2.1 (1) that $b \in f(P)$. Similarly, since $b \in B$, $b = f(r)$, and $r \notin Q$, we know that $b \notin f(Q)$. Since $r \in f(P)$ and $r \notin f(Q)$, then $r \in f(P) - f(Q)$ from Definition 3.3.6. Therefore $f(P-Q) \subseteq f(P) - f(Q)$.

Proof (E).

(1). Let $x \in f(P) - f(Q)$. Then $x \in f(P)$ but $x \notin f(Q)$. By the definition of image, $x \in B$, and $x = f(p)$ for all $p \in P$. But $x \neq f(q)$ for any $q \in Q$.

For convenience, we assign $P - Q = Z$. If $x \in f(Z)$, then $x = f(z)$ for $z \in Z$. We know that for $p \in P$, $p \notin Q$ because that would make $x = f(q)$ true, which we have shown is a false statement. Therefore, $x = f(p)$ for $p \in P - Q$. It follows that $x \in f(P-Q)$. We have shown that $x \in f(P) - f(Q)$ and $x \in f(P-Q)$. Hence, $f(P) - f(Q) \subseteq f(P-Q)$.

(2). First, we prove that if $x = f(a)$ for some $a \in P$ and $a \notin Q$ then $x \neq f(b)$ for any $b \in Q$. This is a proof by contradiction. Suppose $x = f(a)$ for some $a \in P$ and $a \notin Q$, and that $x = f(b)$ for some $b \in Q$. We have reached our contradiction since $x = f(a)$ for some $a \in P$ and $a \notin Q$. This contradicts the fact that $x = f(b)$ for some $b \in Q$. Therefore $x \neq f(b)$ for any $b \in Q$.

We now prove that $f(P-Q) \subseteq f(P) - f(Q)$. Let $x \in f(P-Q)$. We will show that $x \in f(P) - f(Q)$. Hence, $x = f(a)$ for some $a \in P - Q$, by definition of image. It follows that $x = f(a)$ for some $a \in P$ and $a \notin Q$ from the definition of set difference. Hence, $x \neq f(b)$ for any $b \in Q$ by the previous paragraph. Notice that $x = f(a)$ for some $a \in P$ and $x \neq f(b)$ for any $b \in Q$. Thus, $x \in f(P)$ and $x \notin f(q)$ by the definition of image. It follows that $x \in f(P) - f(Q)$ from the definition of set difference. Therefore, we conclude that $f(P-Q) \subseteq f(P) - f(Q)$.

Exercise 8.8.5. [Same as Exercise 5.1.11 (1)] Let A and B be sets, let R and S be relations on A and B, respectively, and let $f: A \to B$ be a function. The function f is **relation preserving** if $x\, R\, y$ if and only if $f(x)\, S\, f(y)$, for all $x, y \in A$.

Suppose that f is bijective and relation preserving. Prove that f^{-1} is relation preserving.

Proof (A). Let $p, q \in B$ such that $p\, S\, q$. Let $m, n \in A$ such that $f^{-1}(p) = m$ and $f^{-1}(q) = n$. Note that $f(m) = p$ and $f(n) = q$. Hence $f(m)\, S\, f(n)$. Since f is relation preserving, thus $f(m)\, S\, f(n)$ implies $m\, S\, n$ for all $m, n \in A$. Thus $f^{-1}(p)\, R\, f^{-1}(q)$. Therefore, if $x\, S\, y$, then $f^{-1}(x)\, R\, f^{-1}(y)$ for all $x, y \in B$.

Suppose $f^{-1}(p)\, R\, f^{-1}(q)$ for some $p, q \in B$. Let $m, n \in A$ such that $f^{-1}(p) = m$ and $f^{-1}(q) = n$. Thus $m\, R\, n$. Since f is relation preserving, thus $m\, R\, n$ implies $f(m)\, S\, f(n)$ for all $m, n \in A$. Observe that $f(m) = p$ and $f(n) = q$, hence $p\, S\, q$. Therefore if $f^{-1}(x)\, R\, f^{-1}(y)$, then $x\, S\, y$ for all $x, y \in B$.

Therefore f^{-1} is relation preserving.

Proof (B). Since f is relationship preserving, we know that $x\, R\, y$ if and only if $f(x)\, S\, f(y)$. So $f(x)\, S\, f(y)$ if and only if $x\, R\, y$, for all $x, y \in A$ and all $f(x), f(y) \in B$. So f^1 is relationship preserving.

Proof (C). Define f^{-1} as the function $g: B \to A$. Let $w, z \in B$. Because g is the inverse of f, then let $w = f(x)$ and let $z = f(y)$. Thus $w = f(x)$ and $z = f(y)$. Because $f(x)\, S\, f(y)$, then $w\, S\, z$. Then $f(w) = f^{-1}(f(x)) = x$ and $f(z) = f^{-1}(f(y)) = y$. Because $x\, R\, y$, then $f(w)\, R\, f(z)$. Therefore if $w\, S\, z$ if and only if $f(w)\, R\, f(z)$. Therefore f^{-1} is relation preserving.

Proof (D). Suppose $f^{-1}: B \to A$. Let $m, n \in B$. Suppose $m\, S\, n$. Because f is both injective and relation preserving for all $x, y \in A$, $f(x)\, S\, f(y)$. Hence, $f(x)$ and $f(y)$ both correspond to elements in B, namely, some $m, n \in B$. Therefore, because $f(x)\, S\, f(y)$ implies $x\, R\, y$, $m\, S\, n$ implies $f(m)\, R\, f(n)$ for all $m, n \in B$. Suppose $f(m)\, R\, f(n)$. Hence, $f(m)$ and $f(n)$ both correspond to elements in A namely, some $x, y \in A$. Therefore, because $x\, R\, y$ implies $f(x)\, S\, f(y)$, $f(m)\, R\, f(n)$ implies $m\, S\, n$. Therefore f^{-1} is relation preserving.

Proof (E). We begin by assuming that $f: A \to B$ is a bijective relation preserving function. Since f is bijective, we know there exists a bijective inverse function $f^{-1}: B \to A$ by Theorem 4.4.5. Since f^{-1} is bijective, by definition we know it is both injective and surjective. Let $p, q \in B$ and suppose that $f^{-1}(p) = f^{-1}(q)$. Since f^{-1} is injective, then $f^{-1}(p) = f^{-1}(q)$ implies that $p = q$ for all $p, q \in B$. Hence $f^{-1}(p)\ R\ f^{-1}(q)$ implies $p\ S\ q$ for all $p, q \in B$. Let $c \in A$. Let $a = b$ for all $a, b \in B$. Since f^{-1} is surjective there exists some a, b such that $f(a) = c = f(b)$. Therefore, $f(a) = f(b)$. Hence $a\ S\ b$ implies $f^{-1}(a)\ R\ f^{-1}(b)$ for all $a, b \in B$. Hence $x\ S\ y$ if and only if $f^{-1}(x)\ R\ f^{-1}(y)$ for all $x, y \in B$. Thus f^{-1} is relation preserving.