Theorem 3.5.6 (**Zorn's Lemma**). *Let $\mathcal{P}$ be a non-empty family of sets. Suppose that if $\mathcal{C} \subseteq \mathcal{P}$ is a chain, then $\bigcup_{C \in \mathcal{C}} C \in \mathcal{P}$. Then $\mathcal{P}$ has a maximal element.*

Proof. We follow [Bro], which says that it adapted the proof from [Lan93].

For each $A \in \mathcal{P}$, let T_A be the set defined by

$$T_A = \begin{cases} \{A\}, & \text{if } A \text{ is a maximal element of } \mathcal{P} \\ \{Q \in \mathcal{P} \mid A \subsetneqq Q\}, & \text{if } A \text{ is not a maximal element of } \mathcal{P}. \end{cases}$$

Observe that T_A non-empty for every $A \in \mathcal{P}$, and hence $\{T_A\}_{A \in \mathcal{P}}$ is a family of non-empty sets. By the Axiom of Choice (Theorem 3.5.3) there is a family of sets $\{F_A\}_{A \in \mathcal{P}}$ such that $F_A \subseteq T_A$ and F_A has exactly one element for all $A \in \mathcal{P}$. For each $A \in \mathcal{P}$, let S_A be the single element in F_A. By the definition of T_A we see that $S_A \in \mathcal{P}$ and $A \subseteq S_A$ for all $A \in \mathcal{P}$; moreover, we have $S_A = A$ if and only if A is a maximal element of $\mathcal{P}$. To prove the theorem, it therefore suffices to find some $M \in \mathcal{P}$ such that $S_M = M$.

Let $\mathcal{R} \subseteq \mathcal{P}$. The family $\mathcal{R}$ is **closed** if $A \in \mathcal{R}$ implies $S_A \in \mathcal{R}$, and if $\mathcal{C} \subseteq \mathcal{R}$ is a chain then $\bigcup_{C \in \mathcal{C}} C \in \mathcal{R}$.

By hypothesis the family $\mathcal{P}$ is closed. Let $\mathcal{M}$ be the intersection of all closed families in $\mathcal{P}$.

We now prove four claims about $\mathcal{M}$. Using these claims, we deduce the theorem, as follows. Let $M = \bigcup_{C \in \mathcal{M}} C \in \mathcal{P}$. Claim 4 says that $\mathcal{M}$ is a chain, and Claim 1 says that $\mathcal{M}$ is closed. It follows that $M \in \mathcal{M}$. Again using the fact that $\mathcal{M}$ is closed, we deduce that $S_M \in \mathcal{M}$. However, we know by Theorem 3.4.5 (2) that $C \subseteq M$ for all $C \in \mathcal{M}$, and hence in particular that $S_M \subseteq M$. As noted above, we know that $M \subseteq S_M$, and we deduce that $S_M = M$, and that is what needed to be proved. (Claims 2 and 3 were not used here, but are needed to prove Claim 4.)

Claim 1. We will show that the family $\mathcal{M}$ is closed.

Let $A \in \mathcal{M}$. Then $A \in \mathcal{R}$ for all closed families $\mathcal{R} \subseteq \mathcal{P}$, and hence $S_A \in \mathcal{R}$ for all closed families $\mathcal{R} \subseteq \mathcal{P}$, and hence $S_A \in \mathcal{M}$. A similar argument shows that if $\mathcal{C} \subseteq \mathcal{M}$ is a chain then $\bigcup_{C \in \mathcal{C}} C \in \mathcal{M}$; the details are left to the reader.

Claim 2. Let $A \in \mathcal{M}$. Suppose that $B \in \mathcal{M}$ and $B \subsetneqq A$ imply $S_B \subseteq A$. We will show that $B \subseteq A$ or $B \supseteq S_A$ for all $B \in \mathcal{M}$.

Let

$$\mathcal{Z}_A = \{C \in \mathcal{M} \mid C \subseteq A \text{ or } C \supseteq S_A\}.$$

We first show that $\mathcal{Z}_A$ is closed.

First, let $D \in \mathcal{Z}_A$. Then $D \in \mathcal{M}$, and $D \subseteq A$ or $D \supseteq S_A$. Because $\mathcal{M}$ is closed, then $S_D \in \mathcal{M}$. Suppose first that $D \subseteq A$. If $D \subsetneqq A$, then by hypothesis on A we deduce that $S_D \subseteq A$, which implies that $S_D \in \mathcal{Z}_A$. If $D = A$, then $S_D = S_A$, and hence $S_D \supseteq S_A$, which implies $S_D \in \mathcal{Z}_A$. Suppose second that $D \supseteq S_A$. Because $S_D \supseteq D$, it follows that $S_D \supseteq S_A$, which implies $S_D \in \mathcal{Z}_A$.

Next, let $\mathcal{C} \subseteq \mathcal{Z}_A$ be a chain. Because $\mathcal{M}$ is closed, we know that $\bigcup_{C \in \mathcal{C}} C \in \mathcal{M}$. There are two cases. First, suppose that $C \subseteq A$ for all $C \in \mathcal{C}$. Then by Theorem 3.4.5 (2) it follows that $\bigcup_{C \in \mathcal{C}} C \subseteq A$, and hence $\bigcup_{C \in \mathcal{C}} C \in \mathcal{Z}_A$. Second, suppose that there is some $E \in \mathcal{C}$ such that $E \nsubseteq A$. Because $E \in \mathcal{Z}_A$, then $E \supseteq S_A$. Because $\bigcup_{C\in\mathcal{C}} C \supseteq E$, it follows that $\bigcup_{C\in\mathcal{C}} C \supseteq S_A$. Hence $\bigcup_{C\in\mathcal{C}} C \in \mathcal{Z}_A$. We deduce that $\mathcal{Z}_A$ is closed.

Because $\mathcal{M}$ is the intersection of all closed families of sets in $\mathcal{P}$, it follows that $\mathcal{M} \subseteq \mathcal{Z}_A$. On the other hand, by definition we know that $\mathcal{Z}_A \subseteq \mathcal{M}$, and it follows that $\mathcal{Z}_A = \mathcal{M}$. We deduce that $B \subseteq A$ or $B \supseteq S_A$ for all $B \in \mathcal{M}$.

Claim 3. We will show that if $A \in \mathcal{M}$, then $B \in \mathcal{M}$ and $B \subsetneqq A$ imply $S_B \subseteq A$.

Let

$$\mathcal{W} = \{A \in \mathcal{M} \mid B \in \mathcal{M} \text{ and } B \subsetneqq A \text{ imply } S_B \subseteq A\}.$$

We first show that $\mathcal{W}$ is closed.

First, let $F \in \mathcal{W}$. Then $F \in \mathcal{M}$, and $B \in \mathcal{M}$ and $B \subsetneqq F$ imply $S_B \subseteq F$. Because $\mathcal{M}$ is closed, we know that $S_F \in \mathcal{M}$. Let $G \in \mathcal{M}$, and suppose that $G \subsetneqq S_F$. It follows that $G \not\supseteq S_F$. By Claim 2 we know that $G \subseteq F$. There are two cases. First, suppose that $G \subsetneqq F$. Then $S_G \subseteq F$. Because $F \subseteq S_F$, it follows that $S_G \subseteq S_F$. Second, suppose that $G = F$. Then $S_G = S_F$, and hence $S_G \subseteq S_F$. We deduce that $S_F \in \mathcal{W}$.

Next, let $\mathcal{C} \subseteq \mathcal{W}$ be a chain. Because $\mathcal{M}$ is closed we know that $\bigcup_{C\in\mathcal{C}} C \in \mathcal{M}$. Let $H \in \mathcal{M}$, and suppose that $H \subsetneqq \bigcup_{C\in\mathcal{C}} C$. If it were the case that $C \subseteq H$ for all $C \in \mathcal{C}$, then it would follow from Theorem 3.4.5 (2) that $\bigcup_{C\in\mathcal{C}} C \subseteq H$, which is not possible. Hence there is some $K \in \mathcal{C}$ such that $K \nsubseteq H$. Because $K \in \mathcal{W}$, then $B \in \mathcal{M}$ and $B \subsetneqq K$ imply $S_B \subseteq K$. By Claim 2 we deduce that $B \subseteq K$ or $B \supseteq S_K$ for all $B \in \mathcal{M}$. Because $S_K \supseteq K$, it follows that $B \subseteq K$ or $B \supseteq K$ for all $B \in \mathcal{M}$. Because $K \nsubseteq H$, it follows that $K \supseteq H$. If $K = H$ then it would follow that $K \subseteq H$, which is not true, and hence we deduce that $H \subsetneqq K$. It then follows that $S_H \subseteq K$. Because $K \in \mathcal{C}$, we deduce that $S_H \subseteq \bigcup_{C\in\mathcal{C}} C$. Hence $\bigcup_{C\in\mathcal{C}} C \in \mathcal{W}$. We deduce that $\mathcal{W}$ is closed.

By an argument similar to the one used in Claim 2, we deduce that $\mathcal{W} = \mathcal{M}$, and therefore we know that if $A \in \mathcal{M}$, then $B \subsetneqq A$ implies $S_B \subseteq A$ for all $B \in \mathcal{M}$.

Claim 4. We will show that $\mathcal{M}$ is a chain.

Let $A, C \in \mathcal{M}$. By Claim 3 we know that $B \subsetneqq A$ implies $S_B \subseteq A$ for all $B \in \mathcal{M}$, and hence by Claim 2 we deduce that $B \subseteq A$ or $B \supseteq S_A$ for all $B \in \mathcal{M}$. Hence $C \subseteq A$ or $C \supseteq S_A$. Because $S_A \supseteq A$, it follows that $C \subseteq A$ or $C \supseteq A$. We deduce that $\mathcal{M}$ is a chain. □

[Bro] Ken Brown, *Zorns lemma: Variation on the proof*, `http://www.math.cornell.edu/~kbrown/434/zorn.pdf`.

[Lan93] Serge Lang, *Real and Functional Analysis*, 3rd ed., Springer-Verlag, New York, 1993.